THE ROLE OF CAUSALITY IN THE CO-ORDINATION OF TWO PERSPECTIVES ON DISTRIBUTION WITHIN A VIRTUAL SIMULATION

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ABSTRACT

Our primary goal is to design a microworld which aspires to research thinking-in-change about distribution. Our premise, in line with a constructivist approach and our prior research, is that thinking about distribution must develop from causal meanings already established. This study reports on a design research study of how students appear to exploit their appreciation of causal control to construct new situated meanings for the distribution of throws and success rates. We provided on-screen control mechanisms for average and spread that could be deterministic or subject to stochastic error. The students used these controls to recognise the limitations of causality in the short term but its power in making sense of the emergence of distributional patterns. We suggest that the concept of distribution lies in co-ordinating emergent data-centric and modelling perspectives for distribution and that causality may play a central role in supporting that co-ordination process.

Keywords: *Distribution; Causality; Randomness, Probability; Variation; Microworld design; Emergent phenomena*

1. TWO PERSPECTIVES ON DISTRIBUTION

Distribution is commonly recognised as one of the key ideas in probability and statistics, certainly at secondary school level. For example, in the UK National Curriculum (DfES, 2000), students at lower and upper secondary level are expected to "compare distributions and make inferences, using the shapes of distributions and measures of average and range." Higher achieving students should be able to extend this to other measures of spread and understand frequency density. The assessment regime in that National Curriculum implies that the above statements refer to distributions of data, either prepared for students or generated through experiments and surveys.

The introduction of digital technology into schools has prompted interest in Exploratory Data Analysis (EDA) as a means of engaging students in statistical analysis, arguably reducing the need for a sophisticated understanding of theoretical statistical principles, demanding an appreciation of probability theory, prior to meaningful engagement. The technology is ideally suited for supporting students as they manipulate data and portray it in a range of different representations in order to infer underlying

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trends. The EDA approach then promotes a perspective on distribution as a representation of collections of actual data, consistent with the goals of the National Curriculum.

Previous research has conceived of distribution as “an important part of learning to look at the data” (Moore, 1990, p. 106) and as an organising conceptual structure with which we can observe the aggregate features of datasets rather than just a collection of individual values (Cobb, 1999; Petrosimo, Lehrer, & Schauble, 2003). Other researchers have claimed the centrality of the concept of data as an aggregate which is characterised by core features that are invisible in any of the individual elements in the aggregate (Konold & Higgins, 2003; Mokros & Russell, 1995). Students, however, have a strong attachment to the case-oriented view; in other words, data are perceived as a collection of individual data values or cases (Wilensky, 1997; Ben-Zvi & Arcavi, 2001). To help students move beyond the case-oriented view, Hancock, Kaput and Goldsmith, (1992) claimed that it is prerequisite for students to mentally construct such an aggregate, before they can see the dataset as a whole. We consider the above approaches to be taking a *data-centric* perspective on distribution. A data-centric perspective on distribution pays attention to the variation and shape of data that has been collected, perhaps through a sampling process.

Petrosino et al. (2003) have suggested that students need to conceive of distribution “as an organizing conceptual structure for thinking about variability located within a more general context of data modelling” (p. 132). Bakker and Gravemeijer (2004), in their attempt to investigate the relationship between data as individual values and distribution as a conceptual entity, examine key aspects of both datasets and distributions such as centre, spread, density, and skewness. They propose a three-level structure; the lowest level comprises of distribution as a set of data values, and the highest level recognises the conceptual entity, distribution. Between these two levels, they position summary statistics such as centre, spread and skewness. They imagine that this structure can be read both *upwards* and *downwards*. In the upward perspective, students tend to perceive data as a series of individual cases, which they can use for calculations of any sample statistics (mean, median, etc.). In the downward perspective, students should look at the data with a notion of distribution as an organising structure, conceiving centre, spread and skewness as features of that distribution. The upward perspective leads to a frequency distribution of a dataset. In the downward perspective, alternatively, theoretically derived distributions, such as the Normal and other probability distributions, are typically used to model data. Bakker and Gravemeijer (2004) chose to deal informally and consistently with core ideas, such as variation and sampling but with distribution still being in a central position. They also envisioned that informal consideration of the shape is the basis for reasoning about distributions. Perusal of recent research literature suggests that reasoning about variation and distributions are strongly associated (Bakker, 2004; Ben-Zvi, 2004; Makar & Confrey, 2003) since “without variation, there is no distribution” (Bakker and Gravemeijer, 2004, p. 149)

However, the notion that variation generates distribution is only part of the story, and it is that part told from the perspective that recognizes what we are terming the data-centric perspective, in which distribution is seen as a collection of data results. Compare this perspective to that of the classical statistician, who accounts for unexplained variation as that part of a hypothetical model which is not apparently associated with a main effect. Here the emphasis is on a model and so we refer to this approach as the modelling perspective on distribution. Indeed, from this perspective, we might reverse Bakker and Gravemeijer’s aphorism to state “without distribution there is no variation.”

When we refer to theoretical distributions (for example, Normal, Uniform and Binomial), we idealise mathematical models, in which we attribute probabilities to a range of possible outcomes (discrete or continuous) in the sample space. In this modelling approach, the model gives rise to variation. Data distributions are seen as variations from the ideal model, the variations being the result of noise or error randomly affecting the signal or main effect, as reflected in the model itself. The modelling perspective on distribution pays attention to randomness and the shape of the probabilities that mould the outcomes, perhaps through some experiment. The modelling perspective reflects, in our view, the mindset of statisticians when applying classical statistical inference. Indeed, Borovcnik (2005) offered six variations on the notion of data being modelled as a main effect together with an error:

(i) Signal + Error	(ii) Pattern + Deviation	(iii) Fit + Residual
(iv) Model + Residual	(v) Explained + Unexplained Residual	(vi) Common + Specific causes

In the same vein, Konold and Pollatsek (2004) viewed the data as a combination of signal and noise, where the signal can be an average value with variation as noise around it. They argued that "the idea of distribution comes into better focus when it is viewed as the distribution around a signal" (p. 171). Bakker (2004), in turn, referred to a second type of signal in noisy processes or shape as a pattern in variability. Bakker (2004) viewed signal as a distribution, such as the shape of a smooth bell curve of the normal distribution, with which we model data. He suggested that the noise in that case is the variation around that smooth curve. The idea of "signal" and "noise" is evident in several research studies (Biehler, 1994; Wild & Pfannkuch, 1999; Noss, Pozzi & Hoyles, 1999).

In our view, a sophisticated view of signal and noise requires a co-ordination of the data-centric and modelling perspectives. We argue that the emphasis in the UK National Curriculum, and indeed as apparent in EDA approaches, is insufficient alone to nurture such co-ordination. We dream of a pedagogy which somehow enables students to appreciate the connection between the data-centric and modelling perspectives on distribution.

The development of such a pedagogy demands that we research the design of tools that aim to facilitate the co-ordination of these two perspectives. Our approach is to adopt a design perspective in which we develop a software-based task to act as a window on thinking-in-change (Noss & Hoyles, 1996). In looking to bootstrap the iterative design process, we found immediate resonance with research on emergent phenomena (Wilensky, 1997), which contain a sense of "organised randomness" (Davis & Simmt, 2003) and a tension between living within rule-defined boundaries and using the space created within those boundaries productively (Johnson, 2001). Thus, we began to think of the challenge of co-ordinating the two perspectives on distribution as one of seeing distribution as an emergent phenomenon (Prodromou, 2004). At the same time, we were alerted to the observation that there is a "centralised mindset" amongst students that may be rooted in a natural habit of interpreting phenomena in a cause-and-effect manner rather than in complex emergent terms (Resnick, 1991; Johnson, 2001; Gould, 2004). However, as we will see, we found that the tendency towards deterministic thinking was a useful resource for co-ordinating the two perspectives.

Our broad aim then is to understand better how students might conceive of data-centric and modelling perspectives of distribution. Furthermore, we aspire to develop environments in which meanings that embrace these two views of distribution might be constructed.

2. EMBRACING CAUSALITY

The modelling and data-centric perspectives on distribution offer different views of variation. In the data-centric perspective, data will spread across a range of values; in the modelling perspective, variation is portrayed as a random movement away from the main effect. In order to co-ordinate these two perspectives, we argue that it is necessary to see them as a duality that encompasses both the deterministic and the stochastic. We therefore examine research on how students apparently perceive the stochastic.

Piaget & Inhelder (1975) suggested in their seminal work that the organism eventually succeeds in inventing probability as a means of operationalising the stochastic. Prior to that achievement, random mixtures were unfathomable and the literature is abundant with examples of how even adults use various, often misleading, heuristics to make judgements of chance (for example, Kahneman, Slovic and Tversky, 1982). How is that process of operationalising the stochastic achieved? Clearly Piaget's constructivist stance would demand that we consider what students already know since therein must lie the resources for coming to appreciate distribution and other key stochastic concepts.

Pratt (2000) reported how students of age 11 years were able to articulate meanings about random phenomena which were remarkably akin to expert-like views in one respect. They understood the unpredictable, uncontrollable and unpatterned nature of randomness. These so-called *local resources* were brought to bear by these students in order to describe short-term randomness. Significantly, these same students were unable to demonstrate meanings for the predictable, controllable and patterned nature of long-term behaviour. Such *global resources* however began to emerge as these students engaged with specially designed tools, in an environment called *ChanceMaker*. This microworld consisted of mini-simulations of so-called gadgets, common random generating devices such as coins, spinners and dice. These gadgets were presented as not working properly and the challenge to the students was to mend them using tools made available within the gadgets. The students began to articulate situated versions of the Law of Large Numbers, such as "the more trials you do, the more even is the pie chart." The significance of this work for the present study is the causal nature of the students' global resources. The number of trials *determines* the state of the pie chart. Pratt (1998) discusses the notion of *phenomenalising*, the process of transforming mathematical ideas into *quasi-concrete objects* (Papert, 1996), which can be manipulated on-screen by the student, who can make sense of the mathematical concept through using it, much as most of us do, to come to appreciate everyday phenomena. By phenomenalising randomness, Pratt claimed that the students were able to exploit well-established knowledge about causality to *concretise* (after Wilensky, 1991) the Law of Large Numbers.

It is our conjecture that, given appropriate phenomenalised tools, students will be able to bridge the modelling and data-centric perspectives of distribution. In this vision, randomness becomes an agent that causes variation and in turn randomness can be "controlled" through parameters, perhaps instantiated as on-screen sliders, that experts might think of as measures of average and spread. Indeed the blurring of a distinction between a measure or representation and a control is one of the hallmarks of using technology to promote using before knowing (Papert, 1996).

It seems though that there is a paradox here. On the one hand, the work of Pratt (2000) makes a prima facie case that technologically-based environments may have the potential to offer a method of constructing meanings for distribution out of causality. On the other hand, such an approach may reinforce the centralised mindset and militate against the construction of distribution as an emergent phenomenon that bridges the data-centric and modelling perspectives of distribution. This apparent paradox lies at the heart of our work. We must design an environment that supports students in discriminating and moving smoothly between data as a series of random outcomes at the micro-level and the shape of distribution as an emergent phenomenon at the macro-level. In that respect, we conjecture that we can build an environment that enables the student (i) at the micro-level to use their understanding of causality whilst at the same time begin to recognise its limitations in explaining local variation, and (ii) at the macro-level to use the parameters as causal agents to appreciate the impact of those sliders on features of the distribution whilst appreciating their failures to completely define the distribution. We intend to use the microworld that embodies these conjectures not only to test those conjectures but further as a window on the evolution of students' thinking about the two perspectives on distribution. Through that window, we ask whether and how students co-ordinate the data-centric and modelling perspectives on distribution.

3. METHOD

Approach and tasks To elaborate this research question, we aimed first to instantiate the conjectures into a microworld that would perturb the students' thinking and act as a window on that thinking-in-change (Noss & Hoyles, 1996). Learning situations are complex ecologies in which many variables interact. Experimental methodologies are often impossible, either because of the confounded nature of the variables or for ethical reasons. We have found in previous work that the delicate process of phenomenalising a mathematical concept in order to observe thinking-in-change demands a gradual sensitising towards that complex ecology. The current ongoing study therefore falls into the category of *design experiments* (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), whereby we gain insights about both thinking-in-change and design issues from the participants' interactions during the iterative design of the microworld. Typically design experiments require several iterations. Each iteration raises new issues about the learning process and generates conjectures about how the design may better help to elaborate the research question.

In this article, we report on pupils' interactions with the third iteration of the microworld. A major issue raised by the first iteration was that the design at that time failed to generate purposeful student activity. In order for it to act as a window on students' thinking-in-change about distribution, it was essential that they were able to explore with relatively little input from the researchers. We therefore searched for a context that might stimulate such activity whilst at the same time encourage focus on distribution as a central concept. In fact, our design strategy has subsequently been influenced by the notions of *Purpose* and *Utility* (Ainley, Pratt & Hansen, 2006). We needed to provide students with a setting that would inspire a deterministic interpretation of behaviour and would find purpose in adopting a perspective that sees behaviour captured and explained in terms of emergent distributions. We have approached the problem by setting the exploration in the second iteration in the context of playing basketball. Our observations during the second iteration alerted us to the significance of the two different perspectives on distribution and the need to find a design that might support their co-ordination.

This article reports on a case study of how two pupils interacted with the third iteration of the microworld, in which they were presented with the basketball-throwing activity depicted in Figure 1. Students were first asked to throw the ball into the basket using various sliders that control the throw. Once this preliminary task was completed, they were asked whether they felt that the simulation was realistic. This normally generated the response that it seemed artificial that the ball was entering the basket every

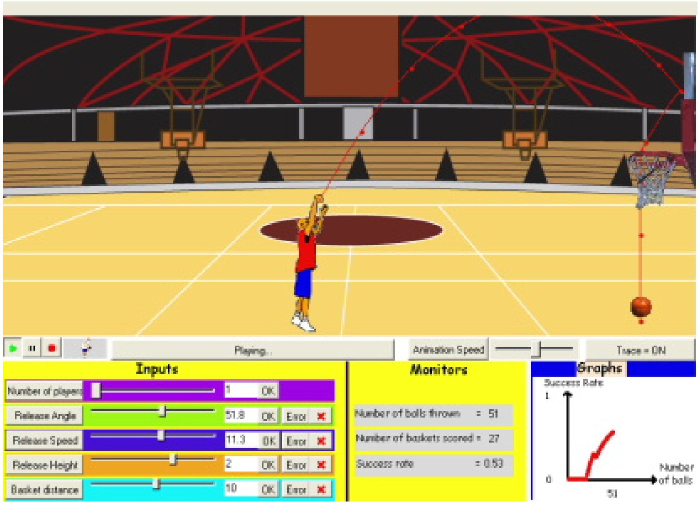

Figure 1. *The player has successfully thrown the ball into the basket. The release angle, speed, height and distance can all be varied using the sliders or by entering the data directly. Once the play button has been pressed, the player continues to throw with the given parameters until the pause or stop button is pressed. The trace of the ball can be switched off. Feedback is shown in the Monitors and Graphs panes.*

time. Since the system was completely determined at this point, the ball replayed faithfully its successful path on every throw. The subsequent discussion typically introduced notions such as skill-level and we showed them the error buttons as in Figure 2 which can make the situation more realistic by allowing for errors in throws.

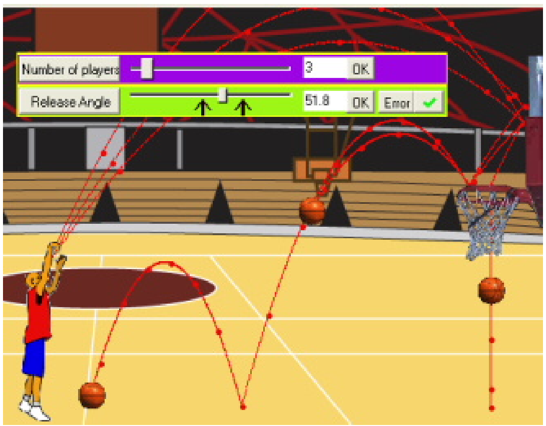

Figure 2. *Three players each throw their ball simultaneously. Because the error button has been pressed, the balls vary their paths. Only one of the three throws is successful. As the three players continue throwing, the release angles will average 51.8 degrees.*

The students were able to control the spread of the error through two arrows on the slider, which corresponded to points that were roughly two standard deviations above and below the mean average as in Figure 2. The students were able to move either or both of these arrows, generating values that corresponded to distributions with differing spreads and bias. The microworld also allowed the students to explore various types of graphs relating the values of the parameters to frequencies and frequencies of success. When the parameters were determined, the graphs appeared as single bar columns as in Figure 3. When the error had been introduced, the graphs appeared as histograms as in Figure 4.

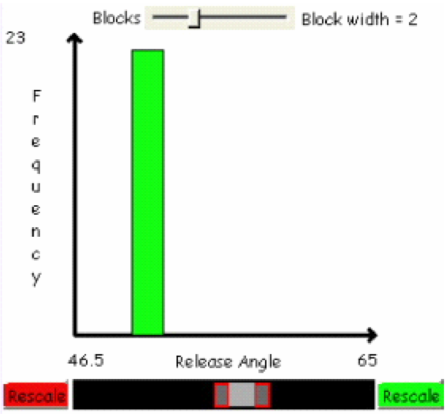

Figure 3. *With no error set, the graph will appear as a single bar. The graphs can be rescaled using the button/sliders.*

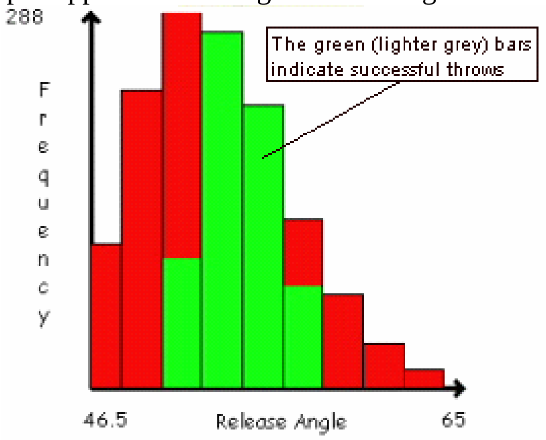

Figure 4. *With error set, the green (lighter) bars in the histogram indicate the successful throws; the red (darker) bars show the remainder.*

(We do not wish to enter here into the debate about whether these graphs should be referred to as histograms. With equal width bars, the matter is of no real consequence.)

Subjects The microworld was trialled with six students in a UK secondary school. The first two pairs of students were, according to their teacher, of average ability. One pair was age 14; another pair was 15. The third pair, 16 years of age, was at an early stage of advanced level study of mathematics and was of above average ability. In this paper we report on the emerging insights of the two 14 year-old students, Tom and Chris. Although we recognize the same issues as reported below from analysis of other pairs, Tom and Chris provide in our view the clearest illustration so far of the co-ordination of the two perspectives on distribution.

Procedure The pair was observed by both authors and the programmer, who had coded the microworld in Imagine Logo, a powerful version of Logo, published by Logotron (www.logo.com/imagine/). For the purposes of this paper, we refer to all three as the researchers. The episode described below took place in one session lasting about 90 minutes. The data collected included audio recording of the students' voices, video recording of the screen output on the computer, and researchers' field notes. The analysis was one of progressive focussing (Robson, 1993). At the first stage, the recordings were simply transcribed and screenshots were incorporated as necessary to make sense of the transcription. Subsequently, the first author turned the transcript into a plain account with no explicit interpretation other than through selection of the more promising sections. The less interesting sections were replaced with discursive descriptions of what happened. At third stage, an interpretative account was written by the first author and discussions about the validity of those interpretations took place with the second author. In this respect, we followed Mason's (1994) advice to make an account of the data before accounting *for* the activity. At the fourth stage of analysis, issues were extracted and turned into conjectures for use in the next and ongoing iteration of the design cycle.

4. FINDINGS

Our analysis suggests that Tom and Chris's meanings for distribution were co-ordinated through four distinctive phases, which we use below to structure the story of how the relationship between causality and variation shifted as they moved through these phases.

4.1. PHASE 1: DETERMINING A SUCCESSFUL THROW

Tom and Chris were introduced in the microworld to a single basketball player who was clearly failing to throw the ball successfully into the net. However, they were shown that his throws could be changed using the various sliders. They were challenged to improve the player's throws.

They began to vary the sliders for release speed and angle as well as height and position. They demonstrated sophisticated intuitions for altering speed and angle in such a way that the path of the ball was gradually moving nearer to the basket. Within two minutes, they had successfully set the sliders to throw the ball into the net (Figure 5). Tom and Chris continued to explore other successful throwing positions by moving the player and finding the corresponding successful release speed and angles. Although Tom and Chris had found an initial successful throw quickly, they appeared to enjoy exploring other values of the parameters that also resulted in a successful throw. It turned out later that it was important that as researchers we allowed Tom and Chris this space to become comfortable with the software, moving the sliders in a playful and exploratory way.

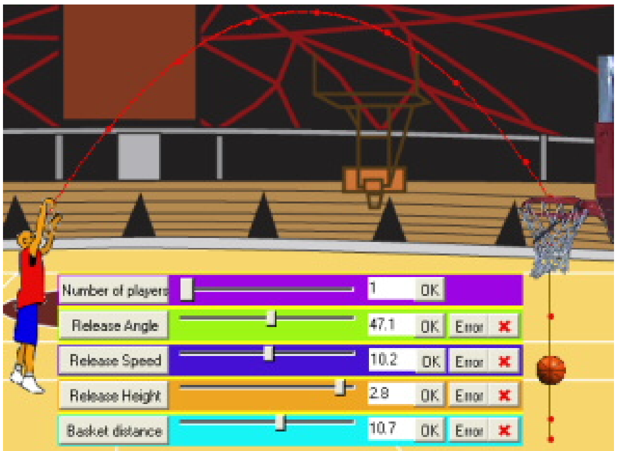

Figure 5. Within two minutes, Tom and Chris had managed to find values for the parameters that caused the player to throw the ball directly into the basket.

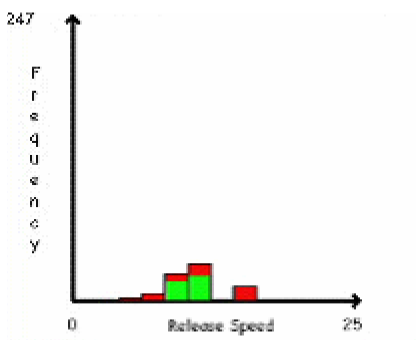

Figure 6. The graph of the release speed, based on 77 throws, shows variation caused by the two boys changing the parameters during the data collection.

Meanwhile, the data for all throws was being continuously collected by the computer since at no stage did Tom and Chris reset the data collection process by stopping the experiment. On two occasions during this phase, the researchers asked the boys to look at the histograms. Although no errors had been introduced, the graphs showed variation. This variation had been created by the manual changing of the parameters during the play (Figure 6). In fact, Tom and Chris did not comment on this variation and the researchers did not probe into the boys' understanding of this aspect. Nevertheless, we suspect that, in the light of the later developments, their own role as agents of variation was an important feature of how they later understood variation in which they were *not* the agents.

4.2. PHASE 2: EXPLORING THE ARROWS

After 24 minutes, the researchers began to introduce the notion of error. It was suggested to Tom and Chris that the previous simulation was not very realistic since, once the correct values had been discovered, the player was successful every time. The boys were introduced to the error buttons. They observed how, when the error button was pressed, two arrows appeared either side of the handle on the corresponding slider. They began to explore the effect of moving these arrows but found it difficult to make sense of what the arrows were doing. (In the transcript, Res refers to the researchers.)

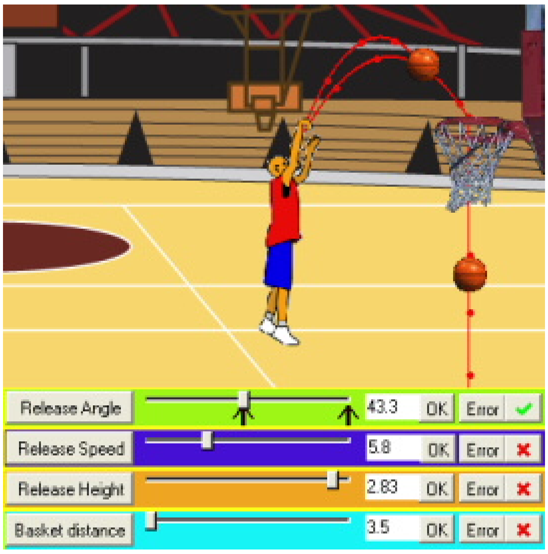

Figure 7. Tom noticed that the path was slightly different even though they had made no changes.

(1) Tom: They *(referring to the arrows)* might help our decision.
(2) *Tom and Chris spent three minutes moving the arrows around whilst the player threw the ball continuously.*
(3) Chris: It doesn't really make it any easier.
(4) Res: Why?
(5) Chris: I can't see how it is improving our chances.
(6) Res: What do you think *(looking at Tom)*?
(7) Tom: Same.
(8) Chris: Still… I'm trying to work out what these arrows are.

At first the boys appeared to associate error with something being wrong, probably associating the word itself and the cross on screen with marking of their work in class. Although confused, the boys continued to try to make sense of the role of the arrows and we did not seek to clarify. After allowing the simulation to run for about two more minutes, Tom noticed that the path of the ball changed.

(9) Tom: It's like… when it's throwing the ball, it's changing occasionally… yeah, like then *(as he was talking the ball took a different path, as in Figure 7)*… it just went differently.
(10) Chris: Could that be to do with the arrows?

A few seconds later, the researchers suggested that they look once more at the graphs.

(11) Res: Can you understand these graphs?
(12) Tom: Quite a lot of angles… spread out.
(13) Chris: We tried a lot of angles. We kept adjusting the angles.
(14) Res: Why?
(15) Chris: I don't know. We were just playing with it.

Tom and Chris were right in that they had indeed been adjusting the sliders during the simulation and that some of the variation in the angles was caused directly by them. However, at the same time, the angles were being chosen randomly by the computer and so some of the variation was due to randomness. It would appear that Tom and Chris recognised variation in this setting where they personally were the agents (lines 13-15) and were possibly entertaining the idea that the arrows may also somehow be involved (line 10).

4.3. PHASE 3: ARROWS AS AGENTS

Half an hour into the session, the researchers suggested that Tom and Chris might begin a new experiment in order to explore more systematically the role of the arrows. They began with error set for release angle but soon introduced error for speed as well (Figure 8).

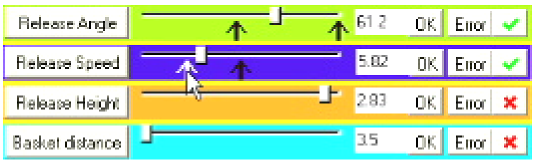

Figure 8. Tom and Chris explored the arrows further by introducing error to angle and then speed.

(16) Chris: The arrows do change it. *(They moved the two arrows closer together)*
(17) Tom: May be… might be between the two arrows… might be…
(18) Res: What might be?
(19) Chris: The release speed... might just be like random.
(20) Tom: It seems to be a bit inconsistent.
(21) Res: Why?
(22) Tom: The shots are like changing… even though that's not changing.
(23) Res: Even though what's not changing?
(24) Tom: The shots are… like that missed *(referring to the simulation in which the throw missed the basket)*… that missed, that one went in *(referring to the following throw)*… but we are not changing them from there *(pointing to the sliders)*.
(25) Res: Not changing the slider. Is that what you're saying?
(26) Tom: Yeah.
(27) Res: But even so sometimes it's missing.
(28) Tom: Yeah.

Our interpretation of this incident is that for the first time, Tom and Chris explicitly recognized that variation could occur without them acting as the agents of change (lines 22-28) and that this insight was accompanied by the preceding recognition that the angles or speeds might be chosen randomly from values between the two arrows (lines 17-20). These two ideas were themselves preceded by an acknowledgement that the arrows did in fact seem to be changing something (line 16). Perhaps such ideas had been gradually growing (line 10). A causal link between the arrows, randomness and scoring or missing seemed now to be postulated. However, such a link was not as yet explicitly established in their minds. Tom and Chris continued to explore, setting errors on and off and moving the arrows around for all of the variables, often simultaneously.

(29) Res: So what conclusions have you got so far about these arrows?
(30) Tom: A bit troublesome.
(31) Chris: Like a random number between the two arrows.

The two boys admitted that they were not yet confident about the idea that the arrows demarcated a region from which a random number would be chosen and so they continued to explore. There was now another intense period of exploration in which they moved the arrows around, sometimes close together, sometimes wide apart, sometimes symmetrically around the handle, sometimes asymmetrically. There did not appear to be much systematicity about this exploration. However, they constantly reviewed the continuing action in the simulation as they tried to make sense of the arrows. Eventually they made a breakthrough.

(32) Tom: If it's close, it's more chance of going in.
(33) Res: What do you mean?
(34) Tom: When the arrows are close together, it's got more of a chance of going into the net.

In lines 32-34, Tom and Chris seemed to have spotted this pattern of behaviour by looking closely at the effect on the animation of moving the arrows. They did not refer to any graphs during this period. A few minutes later however (line 35), they did decide to look at the graphs. Initially, all the histograms appeared to consist of a single bar (Figure 9).

Line 35 is a revealing remark. Chris was content that the graphs apparently revealed no variation since he was still relating variation to changes that they personally had generated (line 35). The causal link relating the arrows to changes in the animation had not been extended to the distribution in the graphs.

The researchers however noticed the small bar to the right of the main bar in the speed graph (see bottom left graph in Figure 9). They helped the boys to rescale the histogram of release speed, at the same time changing the block width (Figure 10).

(35) Chris There's only one because we haven't altered it.
(36) Res: So what does that graph tell you?
(37) Tom: Lines in the middle *(referring to the green (lighter) success bars).*
(38) Res: What about the reds and greens?
(39) Tom: There's an area which is green.
(40) Chris When we get the release speed to… whatever value that is *(pointing to the start of the green area)*… about 11… they're all successes… and when we change it, it's going to miss.
(41) Res: Did you change it?
(42) Chris Did you *(looking at Tom)*?

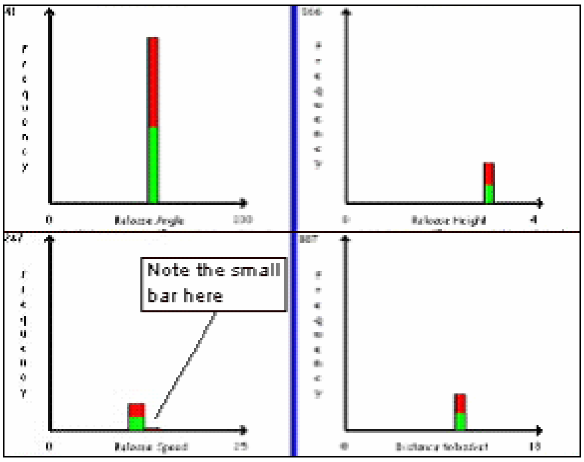

Figure 9. Tom and Chris supposed that the graphs were single bars because they had not made any alterations. They had not yet linked variation in the graphs to variation caused by the arrows. Careful inspection of the speed histogram would have revealed a tiny amount of variation.

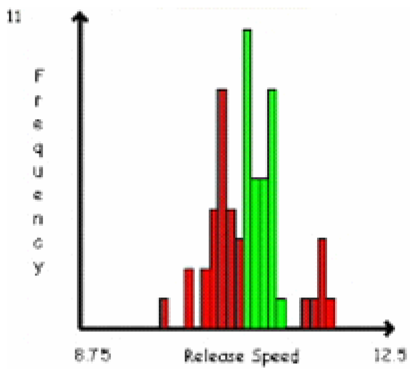

Figure 10. Tom and Chris needed to explain the variation in the histogram even when they had not themselves changed the speed slider.

(43) Tom: Which one?
(44) Chris Speed.
(45) Tom: That was the one with the arrows, wasn't it?
(46) Res: Why have we got different speeds on here?
(47) Tom: Because the arrows change it.
(48) Res: Explain that to him. I'm not sure he understands it yet.
(49) Tom: The arrows make it, like, random. So it's a random number between the two, I think.

It now seemed that Tom had abstracted a causal link between the variation in the histogram and the arrows (lines 47-49). Variation could occur even when they had not acted as agents. Instead the arrows acted as agents through some sort of unknown random mechanism.

One of the difficulties with design experiments is that the researchers are often unable to anticipate activity. Indeed, it is these unexpected outcomes that are often the most influential in shaping the design in the subsequent iteration. Our probing in lines 35-49 was certainly unplanned and as such leaves much unanswered. The precise manner by which Tom and Chris arrived at their conclusion remains a little mysterious. Of course, the strength of design research is that it allows, indeed encourages, such unexpected behaviour and is able to respond later by building such issues into the next design.

4.4. PHASE 4: MODELLING WITH THE ARROWS

Tom and Chris returned to their earlier notion that when the arrows were closer together, the chance of a successful throw was increased. They sought to create a realistic simulation of a player who, perhaps, was not professional but was pretty skilful. They only allowed error on release speed and placed the arrows close together (Figure 11).

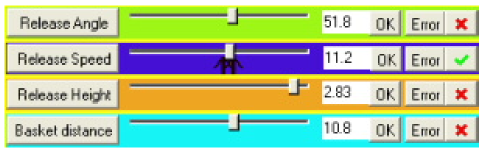

Figure 11. Tom and Chris began to model a skilful but not professional player. They used these values for their parameters.

After 93 throws, they looked at the graphs (Figure 12). The researchers were interested in how the two boys interpreted the histogram of release speed.

(50) Res: When he missed, why did he miss?
(51) Chris Because the speed wasn't enough.
(52) Tom: Most of the reds are at the lower side.

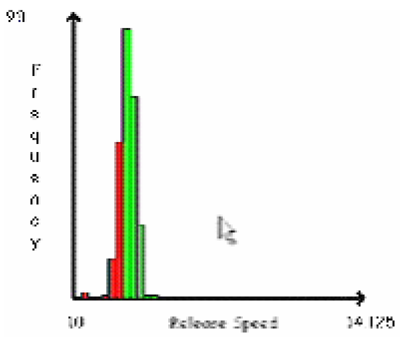

Figure 12. Tom and Chris produced this graph after 93 throws.

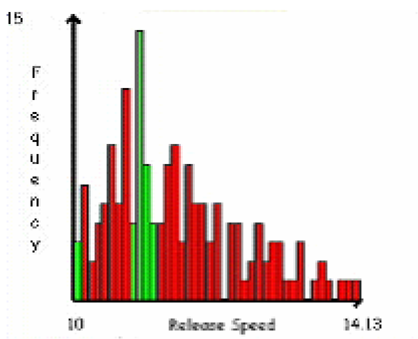

Figure 13. Tom and Chris were surprised to see two green (lighter) areas.

The researchers asked what would have happened if the player had been less skilful. Tom and Chris predicted that the red (darker) and green (lighter) areas would be swapped. It is difficult to understand what they meant by this but we think they meant that the red bars would be higher since there would be more misses (near to the height of the green bars in Figure 12) and the green bars would be lower (near to the height of the red bars in Figure 12). They did not refer to the spread of the graph.

They set the experiment up so that the arrows on the speed slider were wider apart than in Figure 11. After 225 throws, they looked at the graphs. Tom and Chris admitted some surprise at the speed histogram (Figure 13).

(53) Res: Is that what you expected?
(54) Tom: Not really… though there's lots of reds. It's kind of what I expected. I don't know… I don't know… it is kind of what I expected. There's a green there. I don't know why.

The green bar on the leftmost part of the histogram was a surprise to the boys though they were not surprised to see much more red than green since they knew that this player was less skilful. The researchers probed further.

(55) Chris: I think that green area bit will be where he hit off the side *(referring to the more central green area)* and that *(referring to the single green bar)* will be where he got it in straight away.
(56) Res: Ah, so that's why there are two separate areas of green, and why do you think the higher one is where he hit the backboard?
(57) Chris: Because there's more of it to hit… more area.

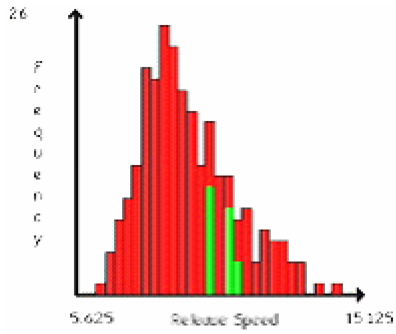

Figure 14. Did Chris's prediction that the results would be more to the left fit the speed histogram shown here?

Chris was able to explain the bimodal green distribution in terms of what he had witnessed in the simulation. The larger spread had in fact allowed the possibility of two distinctive ranges of value of speed that would generate successful throws. (Since then the researchers have found that it is possible to create situations in which there are four distinctive success regions since occasionally it is possible to score after the ball has bounced on the floor!)

5. DISCUSSION

5.1. SUMMARY

We should like to begin by summarising the four phases described above. In phase 1, Tom and Chris worked in an entirely deterministic fashion. They were comfortable with the idea that the path of the ball would be affected by the velocity and position of the throw. They were at ease in changing those parameters in order to determine a path in which the ball moved directly into the basket. This is evidenced by the brief time (about two minutes) needed to generate successful scores. Subsequently during this phase, Tom and Chris observed variation in the graphs though this was not randomly generated but the result of they themselves changing the values of the variables during the running of the simulation. We believe this appreciation enabled them to create a connexion between the variation in the histogram and causal actions in the simulation. During this phase, the agent of change was, of course, the boys themselves.

In phase 2, we used realism as an excuse to introduce the notion of error. As reported above, there was in fact some confusion over our use of this term as the boys initially expected that the computer would report an error, perhaps the ball would be somehow thrown incorrectly. This confusion though was transient. What was less transient was the sense-making process involved in gaining mastery over the arrows. Tom and Chris explored the arrows non-systematically, changing values of the parameters for variables, which also had error set as on. When they looked at the graphs, they believed that the variation was due to the changes that they had made. In this phase, they did not tend to attribute variation to randomness.

Phase 3 was marked by three key insights. First, they recognised that the throws were being chosen randomly from values between the two arrows. Though this is not exactly correct, since the arrows represent values roughly two standard deviations above and below the mean if the arrows are symmetrically placed around the handle, it is a reasonable understanding of the situation. Secondly and almost simultaneously they saw the arrows as agents of the variation in how the player threw the ball in the simulation. Indeed, they articulated this relationship rather concisely, "When the arrows are close together, it's got more of a chance of going into the net." We see this statement as a fine example of what Noss and Hoyles (1996) have called a situated abstraction, a heuristic that characterises the general behaviour of certain phenomena within a specific system. Thirdly, Tom and Chris were able to connect that relationship to the histograms. They were able to discuss how the variation in the histograms was itself caused by the arrows, thus co-ordinating the causal relationships between the simulation, the arrows and the graphs.

In phase 4, Tom and Chris began to use the co-ordinated understanding of the causal role of the arrows to model distributions. They saw this in a situated way. Their aim was to simulate a professional level player or one who was rather less skilful, and they set about that task by moving the arrows nearer or further away from each other. To their surprise, they encountered a bimodal distribution but were able to explain this in terms of the distinctive ways in which a player might throw the ball into the basket, namely directly or off the backboard.

5.2. CAUSALITY AND THE TWO PERSPECTIVES ON DISTRIBUTION

We began with the conjecture that we would be able to build an environment that enables the student to appreciate the limited explanatory power of causality to capture the essence of local variation. At the same time, we ventured that this environment would allow students to use causality to articulate features of distribution.

In fact, we have demonstrated, in Phases 1 and 2, the potential to use notions of skill-related error in a simulated sports context to perturb thinking away from a deterministic mindset towards one of randomly occurring events. Furthermore, in Phases 3 and 4, we have demonstrated how the sliders and arrows can become agents of change, in effect replacing the human agent. Moving the slider changes the position of the distribution. Moving the arrows changes the spread of the distribution. These ideas are articulated in situated ways such as "when the arrows are close together, it's got more of a chance of going into the net." We note the deterministic nature of this situated abstraction. While at the micro-level, causality is shown to have limited explanatory power, at the global level, causality can be harnessed to articulate the relationship between the parameters in the model (average, spread) and the shape of the distribution.

We regard this paradox of seeing the limitations of causality at one level while recognising its power at another level is at the heart of co-ordinating the two perspectives on distribution. We asked whether and how do students co-ordinate the data-centric and modelling perspectives on distribution. We are not able at this stage of our research to elaborate this aim to our complete satisfaction but we believe that insights into the role of causality are significant. We conjecture that as students pay attention to the sliders and arrows they are considering the modelling perspective on distribution, though of course they would not articulate it in that way. The aphorism "without distribution, there is no variation" regards distribution as the agent of variation and in effect comments on the ontology of distribution. We have tried to instantiate that perspective on distribution in the virtual world by offering students direct manipulation over the generational powers of distribution through instantiations of average and spread. In contrast, when students pay attention to the emerging data, they are considering the data-centric perspective on distribution.

In this sense, we find support in this study for the model proposed by Bakker and Gravemeijer (2004) in which distributions can be read from or towards the collection of data. However, whereas Baker and Gravemeijer refer to the student moving bi-directionally between the collection of data and a conceptual entity, we portray the journey as between a modelling and data-centric perspective. For us, the conceptual lies in the co-ordination of these two perspectives. Indeed, we wish to emphasise the equal status of those two perspectives.

Nor is this difference in emphasis merely playing with words. We believe it has teaching implications. There is much excitement about EDA as a modern method for exploring statistics. We share much of that excitement. However, the approach places emphasis upon a data-centric perspective and so far has not offered a coherent statement about how students might abstract from that perspective a rich concept of distribution, which co-ordinates both modelling and data-centric perspectives. Our studies are suggesting that the modelling perspective may need to be given equal status if such a co-ordination is to be encouraged.

Our data so far suggest that causality may be acting as the co-ordinating agent since, not only is it an idea that feels comfortable to students, but it also plays a critical role in helping them to make sense of the relationship between the parameters of the model and the shape of the data. If our ongoing design research continues to support this finding (which in the spirit of design research now becomes a conjecture to be tested in the next iteration), it throws new light on earlier research. The claims that centre around the role of causality in making sense of the stochastic are to some extent out of line with common thinking, which tends to make clear and distinct separations between the two. For example, Piaget and Inhelder (1975) portray randomness as inconceivable within operational thinking, at least until resolved by the invention of probability at a later stage of development. One of consequences of phenomenalisation (Pratt, 1998), turning mathematical ideas into quasi-concrete objects (Papert, 1996), is that mathematical concepts can be expressed in causal terms through the use of situated abstractions, as we have seen in this study. Even it seems statistical ideas, apparently separated from the deterministic world, are accessible to some extent through causal meanings.

We believe the role of causality in bridging the two perspectives on distribution may also have teaching implications. Fischbein (1975) has proposed that some of the difficulty that students have with probabilistic ideas is at least reinforced by a curriculum that emphasizes the deterministic. Indeed, commenting on this assertion by Fischbein, Langrall and Mooney (2005) state, "Children (as well as adults) need to recognize that situations involving chance can be examined and described logically and rationally" (p. 115). If it is true that causality plays the role we are suggesting, the key may lie not so much in reducing the emphasis on determinism as harnessing the power of causality towards the teaching of probability, perhaps through the use of technology.

5.3. LIMITATIONS

The reader must consider the limitations of this research to elaborate the conjectures and research questions. We have reported in some detail only on one pair of students, the clearest illustration of the emerging ideas. Even had it been possible to elaborate the activity of all three pairs, the findings must be regarded as tentative and in a sense interim. We may have, of course, further evidence after the following iteration but design research does not always follow such a smooth path.

It is quite feasible that the role of causality is directly linked to the virtual nature of the setting for this study. Perhaps it is only possible, or at least far simpler, to instantiate these ideas in a technological environment where it is possible to phenomenalise mathematical notions. It is reasonable then to suppose that access to the ideas is understood through the manipulation of the mathematical concepts as articulated through situated abstractions that link causally the inputs and outputs on screen. This is a limitation in so far as we can no more claim that our findings relate to the co-ordination of the two perspectives on distribution in other settings than can other researchers, who unavoidably work in particular settings, though sometimes ill-advisedly in our opinion, ignore the critical role of setting in abstracting. (See Pratt & Noss, 2002, for detailed elaboration of this issue.)

5.4. IMPLICATIONS FOR FURTHER RESEARCH

We have discussed data from a fairly early stage of our work-in-progress. Although we have moved through two previous iterations in order to reach this design, we recognise there are some further design changes to be made. Nevertheless, we believe our results so far indicate support for our conjecture that it is possible to design an environment in which students' well-established causal meanings can be exploited to co-ordinate data-centric and modelling aspects of distribution. Tom and Chris began to appreciate how not only might they themselves be agents of variation, but also how randomness, instantiated in the form of the quasi-concrete arrows, can create histograms in which variation is apparent. In this sense, randomness might become understood as reality once removed. What we have called "letting go of determinism" might be seen as delegating control to a quasi-concrete object that exercises that power through random effects.

It is in the nature of design research that the researchers gradually become sensitised to the ecology of the domain being investigated. We now feel that we have gained a handle on how to support the use of causal meanings in understanding distribution. In that respect we are close to having a design which can be used systematically to test out that conjecture.

- We shall remove the confusion introduced by the term, *error*. In the next iteration we shall simply refer to the arrows and explore what the students make of their role.
- We shall explore in more detail the notion of agency. We expect that agency will become an analytical category varying at least across human, slider and arrows.
- We intend to introduce a graphical representation of the modelling distribution accessed by clicking on the relevant variable such as release angle or speed. We conjecture that access to both the modelling distribution and the data-centric

distribution will enable us to explore more systematically some of the issues described above that still appear relatively mysterious.

- The introduction of a graphical representation of the modelling distribution allows us to introduce a new form of agency. We will hope to allow the students the facility to edit the modelling distribution as a means of transforming the modelling distribution directly but the data-centric distribution indirectly. We ask how will students articulate the chains of agency and how will that impact their co-ordination of the two perspectives on distribution.

Thus, the above outlines our own research programme for the near future. There are however important research questions which our programme will not address. In raising the idea that causality may be a significant agent in constructing a bridge between the data-centric and modelling perspectives, we acknowledge at the same time the possibility that technology is playing a key role in this process. There is fascinating research to be done in exploring the role of causality when other materially-based methods of supporting the co-ordination of the two perspectives on distribution are deployed.

There is much current interest (for example, Pfannkuch, 2005) in researching informal inference. (Informal inference is to be the focus of the fifth conference on Statistical Reasoning Thinking and Literacy to be held at the University of Warwick, August 2007.) EDA is developing interesting pedagogic approaches towards informal inference but we ask whether students can develop an appreciation of the robustness or power of their inferences without constructing a modelling perspective alongside their data-centric perspective. We see this question as one that should tax researchers of informal inference.

REFERENCES

Ainley, J., Pratt, D. and Hansen, A. (2006). Connecting engagement and focus in pedagogic task design, *British Educational Research Journal*. *32*(1), 21-36.

Bakker, A. (2004). Reasoning about shape as a pattern in variability. *Statistics Education Research Journal*, *3*(2), 64-83.
[Online: www.stat.auckland.ac.nz/~iase/serj/SERJ3(2)_Bakker.pdf]

Bakker, A., & Gravemeijer, K. (2004). Learning to reason about distribution. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 147-168). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Ben-Zvi, D. (2004). Reasoning about variability in comparing distributions, *Statistics Education Research Journal*, *3*(2), 42-63.
[Online: www.stat.auckland.ac.nz/~iase/serj/SERJ3(2)_BenZvi.pdf]

Ben-Zvi, D., & Arcavi, A. (2001). Junior high school students' construction of global views of data and data representations. *Educational Studies in Mathematics*, *45*(1-3), 35-65.

Biehler, R. (1994). Probabilistic thinking, statistical reasoning, and the search for causes- Do we need a probabilistic revolution after we have taught data analysis? In J. Garfield (Ed.), *Research papers from ICOTS 4*, Marrakech, Morocco, July, 1994, (pp. 20-37). Minneapolis: University of Minnesota.

Borovcnik, M. (2005). Probabilistic and statistical thinking. *Proceedings of the Fourth Conference of the European Society for Research in Mathematics Education*. Sant Feliu de Guíxols: Spain.

Cobb, P. (1999). Individual and collective development: The case of statistical data analysis. *Mathematical Thinking and Learning*, *1*(1), 5–43.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research, *Educational Researcher*, *32*(1), 9–13.

Davis, B., & Simmt, E. (2003). Understanding learning systems: mathematics education and complexity science. *Journal for Research in Mathematics Education, 34*(2), 137-167.

DfES. (2000). Mathematics: The National Curriculum for England. London: Department for Education and Employment.

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*: Dordrecht, The Netherlands: Reidel Publishing Company.

Gould, R. (2004). Variability: One statistician's view. *Statistics Education Research Journal*, *3*(2), 7-16.
[Online: www.stat.auckland.ac.nz/~iase/serj/SERJ3(2)_Gould.pdf]

Hancock, C., Kaput, J. J. & Goldsmith, L. T. (1992). Authentic inquiry with data: Critical barriers to classroom implementation. *Educational Psychologist, 27*(3), 337-364.

Johnson, S. (2001). *Emergence: The connected lives of ants, brains, cities, and software.* New York: Scribner.

Kahneman, D., Slovic, P., & Tversky, A. (1982). *Judgement under uncertainty: Heuristics and biases*. Cambridge, UK: Cambridge University Press.

Konold, C., & Higgins, T. (2003). Reasoning about data. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 193-215). Reston, VA: National Council of Teachers of Mathematics (NCTM).

Konold, C., & Pollatsek. A. (2004). Conceptualizing an average as a stable feature of a noisy process. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 169-199). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Langrall, C. & Mooney, E. S. (2005). Characteristics of elementary school students' probabilistic reasoning. In G. A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 95-119). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Makar, K., & Confrey, J. (2003). Chunks, clumps and spread out: Secondary pre-service teachers' notions of variation and distribution. In C. Lee (Ed.), *Proceedings of the Third International Research Forum on Statistical Reasoning, Thinking and Literacy (SRTl-3).* [CD-ROM, with video segments] Mount Pleasant, Michigan: Central Michigan University.

Mason, J. (1994). Researching from the inside in mathematics education - Locating an I - you relationship. In J.P. da Ponte & J.F. Matos (Eds.), *Proceedings of the Eighteenth International Conference for the Psychology of Mathematics Education*, Vol. 1 (pp. 176-194), University of Lisbon, Portugal.

Mokros, J., & Russell, S. J. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education, 26*(1), 20-39.

Moore, D. S. (1990). Uncertainty. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 95-137). Washington, DC: National Academy Press.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. London: Kluwer Academic Publishers.

Noss, R., Pozzi, S., & Hoyles, C. (1999). Touching epistemologies: Meanings of average and variation in nursing practice. *Educational Studies in Mathematics, 40*, 25-51.

Papert, S. (1996). An exploration in the space of mathematics educations. *International Journal of Computers For Mathematical Learning, 1*(1), 95-123.

Petrosino, A. J., Lehrer, R., & Schauble, L. (2003). Structuring error and experimental variation as distribution in the fourth grade. *Mathematical Thinking and Learning, 5*(2&3), 131-156.

Pfannkuch, M. (2005). Probability and statistical inference: How can teachers enable learners to make the connection? In G. A. Jones (Ed.), *Exploring probability in school: challenges for teaching and learning* (pp. 267-294). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Piaget, J., & Inhelder, B. (1975). *The origin of the idea of chance in children.* (L. Leake Jr., P. Burrell, & H. D. Fishbein, Trans.). New York: Norton. (Original work published in 1951).

Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education, 31*(5), 602-625.

Pratt, D, (1998). *The construction of meanings IN and FOR a stochastic domain of abstraction*. Unpublished doctoral dissertation, Institute of Education, University of London.

Pratt, D., & Noss, R. (2002). The micro-evolution of mathematical knowledge: The case of randomness*. Journal of the Learning Sciences, 11*(4), 453-488.

Prodromou, T. (2004). Distribution as emergent phenomenon. *Proceedings of the British Society for Research into Learning Mathematics*, *24*(1), 49-54.

Resnick, M. (1991). Overcoming the centralized mindset: Towards an understanding of emergent phenomena. In Harel & S. Papert (Eds.), *Constructionism* (pp. 205-214). Norwood, NJ: Ablex Publishing Corp.

Robson, C. (1993). *Real World Research*. Oxford: Blackwell.

Wild, C., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review, 67*, 223-266.

Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematics education. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 193–204). Norwood, NJ: Ablex.

Wilensky, U. (1997). What is normal anyway? Therapy for epistemological anxiety. *Educational Studies in Mathematics, 33*, 171-202.

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