WHAT'S MISSING IN TEACHING PROBABILITY AND STATISTICS: BUILDING COGNITIVE SCHEMA FOR UNDERSTANDING RANDOM PHENOMENA

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ABSTRACT

Teaching probability and statistics is more than teaching the mathematics itself. Historically, the mathematics of probability and statistics was first developed through analyzing games of chance such as the rolling of dice. This article makes the case that the understanding of probability and statistics is dependent upon building a "mature" understanding of common random phenomena such as the rolling of dice or the blind drawing of balls from an urn. An analysis of the verbalizations of 24 college students, who interact with random phenomena involving the mixture of colored marbles, is presented, using cognitive schema to represent the subjects' expressed understanding. A cognitive schema representing a mature understanding is contrasted to a diversity of observed immature understandings. Teaching to explicitly build the mature cognitive schema is proposed.

Keywords: *Statistics education research; Teaching and learning probability and statistics; Schema learning and instruction*

1. INTRODUCTION

Teaching probability and statistics is more than teaching the mathematics itself. This article makes the case that the development of the understanding of probability and statistics is dependent upon building a "mature" understanding of common random phenomena, such as the rolling of dice or the blind drawing of colored balls from an urn. To make this case, first, the close relationship between the understanding of common random phenomena and the understanding of probability and statistics, is illustrated through historical perspective and educational context. Next, the psychological construct known as a *cognitive schema* is introduced, and a cognitive schema representing a mature understanding of common random phenomena is introduced. Then, a study is presented in which college students interacted with random phenomena involving the mixture of colored marbles, and their verbalizations expressing their understanding of the phenomena are analyzed for indication of mature understanding of the random phenomena as represented in the cognitive schema, or immature understanding. Finally, implications for the teaching of probability and statistics are identified.

2. ON THE UNDERSTANDING OF RANDOM PHENOMENA, PROBABILITY, AND STATISTICS

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2.1. HISTORICAL PERSPECTIVE

Historically, the mathematics of probability and statistics was first developed through analyzing popular games of chance such as involving the rolling of dice. The mathematics emerged 1650-1718, when mathematicians of those days (Pascal, Fermat, Huygens, James Bernoulli, de Moivre), through posing and solving problems related to dice games and other games of chance and gambling, first introduced the concepts of quantified expectation and probability, and related elementary mathematics (David, 1962). Given the pervasive presence in human culture of such random phenomena as the rolling of dice, over millennia, dating back to the times of the Greek and Roman civilizations, David and other authors on the history of probability and statistics (Hacking, 1975/2006; Gigerenzer et al., 1989) have noted that it is remarkable how late in our history that the mathematics of probability and statistics emerged.

The late emergence of the mathematics provides an indication of its relative difficulty to understand. Consider, in contrast, that the origins of geometry date back to the Greek geometer Euclid (about 300 BCE), and origins of algebra date back to the Persian algebraist al-Khwarizmi (about 830 CE). For even the great thinkers of the past, the understanding of random phenomena that would lead to the emergence of probability and statistics was elusive. Thinking of such random phenomena in a new way was key to developing the mathematics. In this article, I use the phrase *mature understanding of a random phenomenon* to refer to a view of a random phenomenon as reflecting or modeling concepts and principles of probability and statistics, that is, as having particular mathematical characteristics; for example (at the most basic level), seeing the rolling of a single die as having six possible outcomes, each with a probability of 1/6 of occurring on any trial.

For 17th century mathematicians, attaining a mature understanding of common random phenomena was instrumental to the development of the mathematics of probability and statistics. Similarly, for students, attaining a mature understanding of common random phenomena is instrumental to the learning of probability and statistics.

2.2. EDUCATIONAL CONTEXT

Understanding mathematics is accomplished through seeing how the mathematics applies to concrete examples and circumstances. This is the principle behind the practice of using physical manipulatives to support students' learning of mathematical concepts, for example, using physical objects to illustrate arithmetic operations such as addition, and using squares and cubes to illustrate area and volume, respectively. For the mathematics of probability and statistics, common random phenomena such as the rolling of dice or the blind drawing of colored balls from an urn are the counterpart to manipulatives to use to facilitate learning. However, the mathematical characteristics of these common random phenomena are not represented as directly within the phenomenon as is the case for the count of a set of physical objects or the measured dimensions of a geometric form. For example, for the rolling of a die, the mathematical characteristic of *probability* of an outcome is not a directly observed physical attribute of the phenomenon. Rather, the idea of quantified probability is indicated through the operation of the phenomenon over time, which has physical characteristics that contribute to the uncertainty and equal probability of each of the possible outcomes. The outcome history unfolding over time shows the frequency and sequencing of the possible outcomes, also reflecting the uncertainty and equal probability of the possible outcomes. In summary, common random phenomena embody their mathematical characteristics (such as probability) indirectly and abstractly, contributing to the difficulty in seeing how random phenomena embody the mathematics, as well as the difficulty of learning the mathematics. A mature understanding of a random phenomenon includes awareness of how the phenomenon embody the mathematics of probability and statistics, and uses that understanding in describing and reasoning about the phenomenon.

3. A COGNITIVE SCHEMA FOR MATURE UNDERSTANDING OF RANDOM PHENOMENA

A schema is a construct in cognitive psychology pertaining to the mental representation of conceptual knowledge, which was formalized with the coming of the era of cognitive science (Minsky, 1975; Rumelhart, 1980), earlier having been introduced through the work of J. Piaget (1926) on cognitive development and the work of F. Bartlett (1932) on cognition and memory. A schema organizes the characteristics or attributes associated with a concept into an integrated whole in memory, and is used in cognitive processing such as recall, recognition, reasoning, and decision-making. For example, a person's schema for "car" may include characteristics of appearance, speed, composite materials, maintenance needs, cost, how to start it, and so on. A schema (e.g., "car") may have subschema (e.g., "sedan" or "hybrid car"), hierarchically related to the more general schema.

Regarding schema and educational practice, research results have shown that conceptual understanding (represented as schema) has top-down influence on reading comprehension, and reading is not just a matter of bottom-up construction of meaning from letters and words; these results have been applied to the design of reading instruction. See Anderson and Pearson (1984) for a review of research in this area. In the domain of mathematics, some studies have found that instruction in schema for categories of situations that appear in mathematical word problems improves student solution performance. See Powell (2011) for a review of such research. The focus in this article, also, is on schema representing real-world phenomena (including their mathematical characteristics), which, through their relation to real-world phenomena, serve to support the application of mathematics to solve real-world problems. In contrast, the focus in this article is not on schema for abstract mathematical concepts that are unbound to real-world context, although such instances of schema are also present in the mathematics education research literature.

In this article, I apply the construct of schema to the concept of "random phenomena" as a means: to formally describe a mature understanding of random phenomena; to illustrate the relative complexity and abstractness of the schema; to support analyzing students' understanding; to clarify teaching objectives regarding probability and statistics; and to identify directions for instructional improvement. Also, the schema provides a view of randomness that unifies "process" (mechanism) and "product" (outcome sequence) aspects, which have been presented as opposing perspectives of randomness in literature reviews over the years (Lopes, 1982; Nickerson, 2002).

A schema representing a mature understanding of a random phenomenon appears in Figure 1 (in a relatively compact readable format). The schema organizes characteristics of a random phenomenon into categories: the physical mechanism producing outcomes, the outcome sequences, and the predictability of the outcomes. Related mathematics (in italics) is integrated into the schema, including probability, expected frequency, and variation. The schema, representing "mature" understanding (within a delimited/basic scope of knowledge), can be expanded to include additional mathematical knowledge, for example, how to enumerate the sample space for sequences of *m* trials, and/or how to calculate the probability of outcome categories derived from the basic set of equally probable outcomes (e.g., the probability of obtaining an even number for the roll of a single six-sided die).

Particular kinds of common random phenomena, including the rolling of a fair die, are subschema to this schema.

In the schema, the characteristics are interrelated and integrated into a whole, and readily available to apply in describing and reasoning about random phenomena. The schema is developed over time, not merely from being told, but from having experiences demonstrating the characteristics that establish them firmly in mind. Some characteristics (1c, 2d, 2e, 3d, 3e, 3f) are included to suppress other naturally occurring ideas such as belief in "being lucky" (having a personal attribute that gives one enhanced performance in correctly predicting) and the gambler's fallacy, which are misconceptions from a mature perspective. The schema, by integrating information into a coherent whole centered around a real-world phenomenon (that reflects more than just mathematics), supports learning and retaining the mathematics, and applying the mathematics to reasoning about random phenomena in the real world.

A common random phenomenon has:	
1. A physical mechanism, with a method to run repeated trials that each produce an outcome:	a. The *mechanism* has *features* that ensure no bias in favor of any particular outcome b. There is *a set of possible outcomes* for each trial, that set numbering more than one (=n); and each possible outcome has equivalent possibility, equal potential, equal *probability* (=$1/n$) to occur on each trial c. Outcomes on successive trials are independent, generated by the same mechanism, which is stable over time
2. Outcome sequences:	a. Over the long run (m trials), each of the possible outcomes has equal *expected frequency* in the outcome sequence (=m/n) b. There is *variation* in the frequency and pattern of occurrence of the possible outcomes among outcome sequences c. Over the long run, outcome sequences show no systematic order or pattern, and are usually mixed-up looking d. The probability of the next outcome in a sequence is independent of past outcomes, even when there has been an unusual sequence of outcomes such as a long streak of a single outcome category e. Orderly/patterned sequences are possible to occur by chance as *rare events,* as they are in the *set of all possible outcome sequences*
3. Predictability (by self/ others):	a. Don't know which outcome will occur, it could be any of the possible outcomes; so difficult to predict b. By chance, no matter which outcome one predicts, one has *probability* of prediction success =$1/n$ c. Over the long run (m trials), *expected prediction success* is m/n times, or $1/n$ of the time; and expect *variation* in prediction success across trials d. Long streaks of prediction success or failure are possible by chance as *rare events*, because such events are in the *set of all possible prediction results* e. Particular prediction strategies are irrelevant to prediction success f. "Being lucky" is not a causal influence on prediction success

Figure 1. A cognitive schema for mature understanding of common random phenomena

A STUDY ON THE UNDERSTANDING OF RANDOM PHENOMENA

3.1. STUDY DESCRIPTION

The subjects (Ss) in the study were 24 undergraduates at the University of Pennsylvania, aged 18-19 years old, 12 male and 12 female. Ss interacted with a Marble Shaking Machine involving the mixture of equal numbers of red (R), blue (B), and yellow (Y) marbles, depicted in Figure 2. Ss were presented with three identical random phenomena (three trays placed on the marble machine) and asked to play a game predicting successive outcomes with each. The mechanisms were chosen for their clearly unbiased natures (including sampling with replacement), so that Ss would be encouraged to interpret them as being unbiased. Several checks were taken to verify that indeed the Ss viewed the mechanisms as unbiased. In a balanced design, Ss were asked to use a different strategy for predicting outcomes with each phenomenon (tray), chosen from strategies that other Ss had used in previous experiments, namely, guessing a mixed-up looking sequence, a regular pattern, based on mechanism start state, or by daydreaming about the colors. For each tray, the Ss kept a record of the color sequence to fall (for 12 marbles) by inserting colored pegs into a pegboard, and kept track of their prediction success using plastic chips. Ss obtained an average prediction success rate (4 out of 12) for two phenomena, and a high prediction success rate (7 out of 12) for the other. The manipulation of prediction success rate was accomplished through a design feature of the Marble Shaking Machine that allowed the experimenter to surreptitiously determine the color of the next marble to fall, after having heard the S's prediction of color.

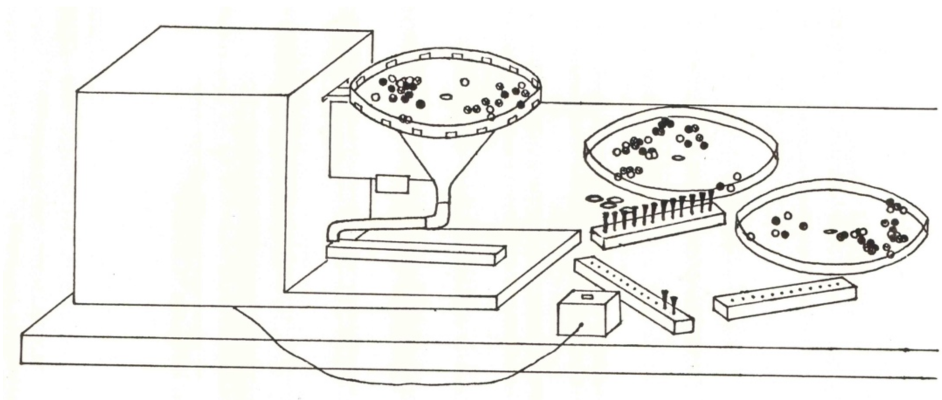

Figure 2. Marble Shaking Machine and supplementary apparatus

At various points in the experimental procedure, Ss were asked to give a numerical prediction of their prediction success (out of 12 trials with a phenomenon, i.e. tray), and to explain why they predicted that number. After experience with the three phenomena (trays), Ss were asked to choose a tray to use for a final round in which they would be paid (25¢) for each correct guess, as well as 25¢ if they accurately predicted their actual total correct predictions. They were also asked to tell which strategy they planned to use for the money-earning round and why. Of interest was how the high success using a particular strategy might influence Ss' judgments and reasoning regarding the phenomena. Near the end of the experimental session, Ss were asked what the word "random" meant to them, and whether the phenomena (trays) were random. A more detailed description of the experimental design and procedure appears in Kuzmak (1983). Results that follow are additional analysis beyond what is reported in Kuzmak (1983).

3.2. RESULTS AND DISCUSSION

In interacting with a phenomenon, Ss' verbalizations of their judgments and reasoning give an indication of their understanding of the phenomenon. In the experiment, Ss' verbalizations in response to questions about their predicted success and why, and about whether the phenomena were random, were analyzed for indication of mature understanding of the random phenomena as represented in the cognitive schema in Section 3, or immature understanding. The questions and features of the experiment (including using different strategies to predict outcomes), along with having the experience of high prediction success for one of the phenomena, served to direct Ss to draw on their knowledge regarding the predictability of the phenomena, which, if mature, would include knowledge in the schema, specifically, 3b, 3c, 3d, and 3e. The basic understanding in the schema 3a (that one does not know which outcome will occur, it could be any of the possible outcomes, so it is difficult to predict) was not specifically the focus of this study, there being evidence of presence of this basic understanding in 5-year-olds and possibly 4-year-olds (Kuzmak & Gelman, 1986).

Mature understanding Four of the 24 Ss (17%) provided verbalizations that were consistent with the schema for mature understanding of random phenomena (hereafter referred to as “the Schema”). These Ss:

- gave a verbal indication that prediction success of 4 is expected by chance (consistent with Schema 3c, given 12 trials and 3 possible color outcomes)
- predicted 4 for prediction success in the final round, despite having obtained high success with a strategy (consistent with Schema 3d, regarding understanding that a long streak of prediction success is possible by chance as a rare event)
- gave a verbal indication that strategy makes no difference to prediction success (consistent with Schema 3e and/or 3b)
- said that “yes” the trays/phenomena were random (3 Ss) or “appear pretty random” (1 S).

For the four Ss, their verbalizations relating to the four bullet points above are provided in the next four subsections, along with the corresponding experimental context and experimenter prompts. The four Ss are labeled as A, B, C, and D for the four sets of verbalizations that follow.

Prediction success expected by chance Near the beginning of the experimental procedure, the Ss were given an initial opportunity to play a short practice game with the Marble Shaking Machine (for six trials with a demonstration tray of marbles) to familiarize them with the operation of the machine and the procedure for the guessing game; they observed a mixture of the three colors of marbles fall, and guessed correctly two times out of six. Then the experimenter questioned them and they responded.

Exptr: If you were to play the game now with this tray here, how well do you think you would do? How many do you think you'd get correct out of 12? … And why do you say <S's chosen number>?

A: … I would say four. I would make four correct guesses. … Because there are three different colors, and I figure I have a one out of three shot for each one.

B: Four out of 12. … Because with the random odds, the three colors, 12 times, the odds would suggest four.

C: Four. … Just because it's a game of chance and the odds would have it that four out of 12 would probably be right.

D: Four. … Well. Twelve guesses, three colors. If I guessed the same color each time, I have a chance of getting, four of them would be right. So I'll guess four. [Later:] On any one guess, the chance of any number, of any color coming up is equal, so I just, first I was planning to guess red throughout. And I figured I'd get one-third right.

These responses show an understanding that, over a 12-trial run, the expected prediction success is 12/3 (=4) times or 1/3 of the time, based on there being three equally likely colors possible for each trial (consistent with Schema 3c).

Prediction success expected after obtaining high success During the experiment, the Ss played the game of prediction with each of the three trays/phenomena, for 12 trials per tray, and experienced a high level of prediction success (7 out of 12) with one of the trays/phenomena. Of interest was how the Ss would interpret the experience of receiving high prediction success. Among the four Ss, as it turned out, each had been assigned to a different experimental condition, so that each experienced the high prediction success while using a different one of the four possible prediction strategies. By the binomial distribution (N=12, p=1/3), the probability of obtaining prediction success as uncommon as 7 correct out of 12 (that is, $x \geq 7$ or $x \leq 1$) is 0.12 for a single 12-trial run; the probability of obtaining such uncommon prediction success on at least one of three successive 12-trial runs is 0.32. Accordingly, given that the experiment involved three successive 12-trial runs, the occurrence of one case of high prediction success is actually not uncommon. After having the experience playing with the three trays/phenomena, the Ss selected one of the trays and a strategy to use for the final money-earning round, and were questioned by the experimenter and responded.

Exptr: How well do you think you'll do when you use that way of guessing [S's chosen strategy to use for the final round]?

A: … I think I'll get four right. … Because it's, I still think I have a one out of three chance. And I've done it [played the game] three times, and two out of the three times – four chances [meaning got four out of 12 right]. … 'Cause I think I just got lucky on the second tray [with high prediction success].

B: Four out of 12. … Well, I considered them all to be random patterns [the outcome sequences for all three trays]. ... I still expect to get four out of 12.

C: About four out of 12. … Just because chances are that's what it's going to come out to. [Earlier, after receiving high prediction success:] Because it's still just chance. I just managed to get lucky.

D: Four.
[Earlier, after high prediction success:] Because I maintain that the thing is still random, and, even though I got seven right this time, that's just chance. It was just not over a long enough run.

These responses show the understanding that a long streak of prediction success is possible to occur by chance as a rare event (consistent with Schema 3d). By continuing to express an expectation of prediction success of four out of 12, after having experienced high prediction success with a prediction strategy with one of the trays/phenomena, the four Ss gave an indication that they attributed the high prediction success to chance, the high success being a possible random occurrence. The four Ss explicitly described their experience of high success in predicting outcomes to be "random," "just chance," and/or "just … lucky." Note that the reference to "lucky" here appears to be with the meaning of a chance occurrence, and not with the meaning of being lucky as a personal attribute having causal influence on prediction success.

Judgment of the influence of prediction strategy on success During their experience playing the game of prediction for the three trays/phenomena, and being asked by the experimenter their predicted success for each tray and why, the Ss explained their repeated prediction to have success of four out of 12 by mentioning the lack of influence of prediction strategies on prediction success.

A: Four. … Because I still see that I have the same chances as I did before, 'cause I'm really not doing anything different. …
Three or four. … 'Cause I don't see that there's any way I could increase it or decrease it. …
I'll get four. … Because I still see that the way that I'm guessing has nothing to do with the results. I don't think one way of guessing has any particular advantage over another one. When I have no control over this. …
Four. … Because it's still, I don't think any particular way that I guess is going to change anything.

B: Four out of 12. … Because there should be no relation to the color of which I'm thinking and the physical apparatus's dropping those marbles. …
Four out of 12. … Because the, with the odds remaining constant, the, whatever happened in past experiments, whatever happened in past results won't affect the current results. …
Four out of 12. … Because the procedure is completely random, independent of previous trials.

C: Four. … Because any way of guessing is reasonable. I mean, and no way will do any better than the other. It's just chance. …
About four. … Because it's still just chance. I just managed to get lucky. …
About four out of 12. … It's still just chance. You can't influence how, what marble will fall down. …
About four out of 12. … Well, for one thing, because I matched [actually guessed four correctly last time], and for another because it's still luck. …
About four out of the 12. … Again, just because of luck. Because patterns don't really mean much.

D: Four. … 'Cause if I'm not paying attention to what is coming down on any one time the marble falls, there's an equal chance of any of the three numbers coming up, colors coming up, so four out of 12 I should be right. …
Four. … 'Cause if I guess randomly, the chance is one out of three that I'll come up with the right answer. So out of 12 guesses, I'll come up with the right answer four times. …
Four. … 'Cause equal chance for any one, I would say. …
I think I'll get four. … Because any pattern that appears is random, so it's just like you're guessing random almost. …
Four. … Because whatever falls down is just random even though I might think it might be in a way the position, so out of 12, one third will be right. …
I think I'd get four. … Because I maintain that this thing is still random, and, even though I got seven right this time, that's just chance. It was just not over a long enough run.

The four Ss gave an indication that they view the particular prediction strategies as irrelevant to prediction success, by explicitly stating that the prediction strategies do not influence prediction success (Ss A, B, C) and/or that the likelihood of each color/ outcome is constant over time regardless of one's prediction strategy (Ss B, D) (consistent with Schema 3e and 3b, respectively).

Judgment of randomness of phenomena Immediately before playing the final money-earning round, the experimenter asked the Ss what "random" meant to them, and whether the phenomena (trays) were random, and the Ss responded. By experimenter error, Subject C was not asked these questions (the only S out of the 24 Ss not asked), but Subject C's responses earlier in the experiment expressed that the phenomena were chance or random.

Exptr: Would you say that these trays here are random?

A: They [the trays/phenomena] all have the same amount of marbles, and all the marbles are the same. Then they're random.

B: On the basis of the fact that the trays seemed to be shaped identically, but, more important, that the, they seem to contain the same number of evenly shaped, evenly weighted marbles, yes, I'd say they're random.

C: [Earlier comments:] … it's a game of chance … It's just chance. … Because it's still just chance. … It's still just chance.

D: Not totally [random]. I would, pretty random. If they're the same number of marbles, and if they're the same size and weight, if the trays aren't tilted, if they start off in a very mixed position. I mean, I don't know if the numbers are large enough, but they appear to be pretty random.

The Ss' responses indicate that they view the phenomena (trays) as random phenomena. Note that three of the Ss refer to features of the physical mechanism for the phenomena (e.g., equal numbers of each color marble) that indicate no bias in favor of any particular outcome (consistent with Schema 1a). By identifying the phenomena (trays) as random, the Ss confirm that their judgments and reasoning about the phenomena reflect their understanding of random phenomena.

Summary In summary, among a college undergraduate population, some students do provide evidence through their verbalizations of having a mature understanding of common random phenomena (for the delimited/basic scope of knowledge identified in the Schema). They express mature knowledge of the predictability of random phenomena, consistent with the Schema 3b, 3c, 3d, and 3e. Knowledge in the schema is readily available to them to apply in describing and reasoning about real-world random phenomena. However, the majority of the sample of college undergraduate students did not show mature understanding, but rather various immature understandings.

Immature understanding Twenty of 24 Ss (83%) provided verbalizations that were not consistent with the schema for mature understanding. Table 1 summarizes Ss' "immature" responses, highlighting ways that their understanding differs from mature understanding. The table reveals that there is a diversity of immature understanding. For example, several Ss (25%) did not indicate that expected prediction success by chance is four or 1/3 of the time; their responses included predicting five ("less than half") or six ("an average number"). Some (62.5%) believed prediction strategies to be effective, while they agreed that the phenomena are random. Some (17%) abandoned the belief that the phenomena are random after having experienced high prediction success, not attributing the high success to chance. Ss who expressed belief that at least one prediction strategy could be effective are identified in the table with the prediction strategy that they chose to use for the final money-earning round.

Table 1. Immature understandings of random phenomena

Immature Understandings (IU)	Number & Percentage of Subjects	Indicated 4 expected by chance (Y/N)	Predicted success for final round: Median (Range)	Are phenomena/ trays random? (Y/N/other)
1. Strategy makes no difference to prediction success; expected success not calculated, but judged from experience	1 (4%)	N	4	Y
2. Gambler's fallacy, predicting mixed-up sequence works	5 (21%)	Y (all)	6 (5-6)	Y (all)
3. Possible to predict without logical cause, using ESP, intuition	4 (17%)	Y(3), N(1)	5 (5-6)	Y (all)
4. Possible mechanism-based way to predict	4 (17%)	Y (all)	5 (5-6)	Y (all)
5. Possible pattern-based way to predict, don't see why it works	2 (8%)	N (all)	6.5 (5-8)	Y (all)
6. Possible regular pattern-based way to predict (falling in twos)	1 (4%)	Y	5	"not conclusively"
7. Predicting regular pattern works (repeat BYR sequence)	1 (4%)	Y	7	"sort of"
8. Predicting based on mechanism works	2 (8%)	N (all)	7.5 (7-8)	N, "hoping ... not"

Each category of immature understanding (IU) in Table 1 (numbered row of table) is described and discussed in subsections that follow, including contrasting the immature understanding with the mature understanding of random phenomena represented in the schema above. To illustrate the categories of immature understanding, verbalizations for nine of the 20 Ss are provided in the sections that follow, with each of the nine Ss uniquely identified with a letter from E to Z. It is beyond the scope of this article to represent the Ss' various immature understandings in schema format, although that could be a focus for further study.

Immature understanding 1 One subject (4%) (labeled E in the following) responded showing some elements of mature understanding, specifically, provided verbalizations indicating prediction strategy had no influence on prediction success (consistent with Schema 3e), and attributed high prediction success to chance or luck (consistent with Schema 3d), as shown, respectively, in the two responses given here.

E: I can't see how anything could determine what would fall next. … I would say going by the order, say, of the marbles as they fell out, to try to make any sort of pattern would be, it wouldn't necessarily be bad, but it wouldn't help you either.

E: [After having high success in prediction] Well. I would say four because I would think that the fact that I got seven this last time was lucky, happened to be chance. It was lucky that I got that many right, and I wouldn't think that I would get that many right if I were to try to guess randomly again. So I would say four 'cause it was a relatively low number.

However, the S showed lack of knowledge concerning calculating the probability of prediction success for a trial (Schema 3b) or expected prediction success in 12 trials (Schema 3c), and instead expressed various expectations, influenced by accumulating empirical results over time:

Exptr: [Before demonstration of the Marble Shaking Machine] How many correct guesses do you think you would make out of twelve?

E: Six. I'd say about six. … Because it's an average number. Half. I'd think I'd get about half right.

E: [Before trying the first tray] Three. … Well, because I guess I'm more conservative in my guesses than last time seeing how difficult it was, and I just say about a quarter of them, I would get.
Exptr: OK, and is there any particular reason why you chose one quarter of twelve?
E: No, just because I figured it was a pretty reasonable amount.

E: [After experience predicting with the first tray] Four. … Because that's what I got when I did it this time, so I figure chances are I'd probably get about the same amount as I did that time.

E: [Before trying the second tray] I'd say three. … Well, because, I would pick a low number because I don't think I'll get many right, and again that just seemed to be a random number. Again, as I said last time, one quarter of them right just seems to be –

E: [After experience predicting with the second tray] I would say, four. … Well, I would say four because I would think that the fact that I got seven this last time was lucky, happened to be chance. It was lucky that I got that many right, and I wouldn't think that I would get that many right if I were to try to guess randomly again. So I would say four 'cause it was a relatively low number.
Exptr: OK. You did say, though, originally, that you would expect to get three.
E: Uh-huh.
Exptr: And you have increased that, but, is there any reason for that or just –
E: Well, I, just that, my chances guessing randomly, might be a little bit higher than three. Four. So that is why I raised it. Just because I had got so many when I guessed this time.

E: [Before trying the third tray] Four. … Because that seems to be a pretty average number to get right, judging, I got around that number correct each time I did it.
Exptr: OK. Um, so that has nothing to do – like last time you said initially three for that way of guessing. Does the fact that you say four this time have anything to do with the fact that this is a different way of guessing, using the positions of the marbles – I mean, you said a different number initially.
E: No, no. The fact that I increased it up to four does not have anything to do with the way, the way I'm guessing. I just think that my chances are probably about the same as they were for the time I guessed three. I just think that I was, chances are I could get more, and I could've gotten more with the three.

Subject E's responses show that some elements of mature understanding may be present without knowledge of calculating probability or expected frequency. Other Ss (see Table 1) show knowledge of calculating probability or expected frequency, but lack other elements of mature understanding.

Immature understanding 2 Five Ss (21%) viewed the phenomena as random, and showed knowledge of the expected prediction success in 12 trials; they expressed belief in the gambler's fallacy, that predicting a mixed-up looking sequence of outcomes (e.g., avoiding predicting streaks of one color) should increase one's chances of predicting correctly; they said that they would use that strategy for the final money-earning round; and they predicted success of 5 or 6 (above average) for that final round (see Table 1). Four of the five Ss were among the six Ss in the experiment who, by the balanced design, had been assigned to the condition of having high success in prediction while using the strategy of predicting a mixed-up looking sequence. Note that if other Ss in the experiment had been similarly assigned, they may have also expressed a belief in the advantage of predicting a mixed-up looking sequence. Therefore, the occurrence of 21% for the sample expressing belief in the gambler's fallacy may be an underestimate for the belief for the population. Regarding people's understanding of randomness, numerous research studies have shown that people who are instructed to generate random sequences have a tendency to produce sequences with more alternations than actually occur for random phenomena, which is consistent with the gambler's fallacy (Nickerson, 2002; Wagenaar, 1972). While the five Ss expressed the immature understanding that predicting a mixed-up looking sequence could increase their likelihood of predicting correctly for a random phenomenon with equally likely outcomes, some of these Ss also expressed awareness that their reasoning didn't make sense. Verbalizations of three of the five Ss (labeled F, G, and K), provided here, illustrate their understanding.

Exptr: What way of guessing were you using that time?
F: I was picking one that we had had least of or we had gone a long period of time without having it. And trying, saying that that was all we were gonna get. I was trying to mix up colors.
Exptr: Now, if perhaps you were to use that same way of guessing the next time, to play another game, how many would you expect you would get right out of 12?
F: Six. … Because, all right, I think it showed something that I got more than four, OK? The method that I was using. So, it's better than just a random probability. But I think that seven was luckier than I would predict that I would get the next time.

Exptr: [Later] And, would you say that any of these trays are random?
F: That the trays are random, or the methods are random?
Exptr: That the way the marbles come out of the trays is random.
F: Yes. Well, each individual time, it's random. But I think that overall you can say that there will be approximately the same number of each marble coming out. This doesn't really make any sense because – (laugh)

Exptr: Just for the record, that time what way of guessing were you using?
G: I was looking at the tees [pegs] in the block, and seeing which ones came out beforehand. So, I, see here, this [pointing to block with record of colors to fall], I would say blue after that. Red, yellow, yellow, I'd say blue next.
Exptr: Now, if you were to use that same way of guessing, with that tray, let's say next time in the money round, how well do you think you'd do? How many do you think you'd get right out of 12?
G: Six. … Seems to work better than the other two methods. And I didn't do too well on the other ones. I did pretty well on this.
Exptr: So you figure, why particularly six as your number?
G: A little conservative. I figured I did seven on this thing. Could be one, around plus or minus one.

Exptr: [Later] And would you say that these trays are random?
G: I would say so. Assuming all the balls are the same amount. They're all weighted the same also.

Exptr: OK. You were using the way of guessing where you guess whichever color hasn't come up.
K: Right.
Exptr: Now, if you were to use that same way of guessing again, how well do you think you would do? How many do you think you'd get right out of 12?
K: Well, intellectually (laugh), it makes no diff, I say it makes no difference, but my gut reaction says that that's more practical, and that you, I'd have a, I'd probably do better.
Exptr: So what would you predict out of – ?
K: I said that I'd predict five out of that one. That was my prediction.

Exptr: And so why did you say five?
K: Because that feels like it's a, a better method. Feels – let's keep it! (laugh)

These Ss lack the knowledge that prediction strategy has no influence on prediction success, and, when faced with the random phenomena, express their belief that predicting a mixed-up looking sequence gives an advantage. Some perceive that this belief doesn't make sense because the outcomes are equally likely, but express the belief nonetheless.

Immature Understandings 3, 4, and 5 Ss showed a belief that other prediction strategies provided an advantage in achieving prediction success for the random phenomena. As summarized in Table 1, four Ss (17 %) expressed the belief that they could predict better without logical cause using ESP (extra-sensory perception) or intuition; four Ss (17%) expressed the belief that they could predict better using a mechanism-based strategy such as guessing a color that starts off close to the central hole before shaking starts; and two Ss (8%) expressed the belief that they could predict better using a strategy of predicting a pattern, although they didn't see why it worked. For some of these Ss, but not all, the strategy that they identified as giving them an advantage for the final money-earning round was the same strategy with which they had previously experienced high prediction success (two out of four for IU 3, three out of four for IU 4, and two out of two for IU 5). Verbalizations of three Ss (labeled N, R, and V), illustrating each of these three categories of belief regarding prediction strategy, are provided here.

Exptr: [For money round] And why do you predict five?
N: Well, I think I'm going to use a combination of the different methods (laugh), so, …
Exptr: And what way are you going to use for guessing?
N: Well, I'll start off guessing, and I won't guess three in a row because that didn't occur at all. Three of the same color in a row. And I think I'll probably guess two of the same color in a row some of the time. I think mostly just a lot of ESP.

Exptr: [Later] And then, would you say that these trays are random? …
N: Yes, I would say so.
Exptr: In other words, so in what sense are they random then?
N: That the balls could come down any way given an infinite number of tests you did, you'd have an infinite number of different combinations of things come down.

Exptr: [For money round] Is there any particular way of guessing that you're going to use with that tray?
R: Probably stick to the first method.
Exptr: OK. So you'll, which way was that?
R: That was, looking at the distribution of the marbles before, you know, the machine was turned on.
Exptr: OK. Now earlier when you mentioned that method, you said that you wouldn't count on, you know, using that method. How do you feel about that now?
R: Well, that was before I heard the other two methods, and, I would say the first one at least has a little bit of science to it, in that you're actually looking at the distribution before deciding, and that can possibly, I would say, give some indication of what will come up, rather than just guessing a color from what colors have come up before.
Exptr: How many do you think you would get right using this way of guessing?
R: I'd say five.

Exptr: [Later] Could you use the word random to describe these trays here?
R: Well, in the sense that it's too difficult to figure out which marble's gonna drop through, then it's random. But in the sense that, if you repeated the experiment the exact same way with the marbles sort of like distributed before you started, and the machine happened to work in the same way, again, then you'd get the same marble, so that

wouldn't be random. But since it's beyond, you know, it is beyond possibility for repeating it exactly, then you could call it random.

Exptr: [For money round] What way of guessing are you going to use?
V: OK. I'm going to try and go in patterns depending on what comes out.
Exptr: OK. And why will you guess that way?
V: Because I got seven doing it before in that way. So, I'll try it again.
Exptr: OK. And how well do you think you'll do, then, using that way of guessing, with that tray?
V: Five.
Exptr: OK. And why do you say five?
V: Because that could've just been luck, but it might not've been since, I mean, it's such a difference. It's three better than either of the other two. So I think it's more than just luck, but on the other hand, it might be. So, I don't want to say six or seven.

Exptr: [Later] Could you describe these trays here as random?
V: The trays with the marbles in them?
Exptr: Right.
V: Yeah, I think so. Although, actually all the colors are – tend to bunch together a little bit. See? But, I think, yeah, they're probably random. The reds and the blues are much more than the yellows, bunching together.
Exptr: The reds and the blues bunch together more than –
V: Except in that one [tray]. Yeah, in these two [trays].
Exptr: OK. So would you say it was random the way the marbles came out of the hole?
V: Not really sure. (laugh) I'm really not sure. They do tend to come out in pairs.
Exptr: OK. Uh, and why would they come out in pairs?
V: Because the colors tend to be bunched together.

These Ss fail to show knowledge that prediction strategy has no influence on prediction success for random phenomena because each outcome is equally likely, or that high success in prediction may occur purely by chance as a rare event. They express beliefs that particular strategies (such as using ESP or intuition, using the initial state of the mechanism, or following patterns) may give an advantage, while still judging the phenomena to be random. Some express uncertainty in their judgments regarding predictability, but judge so nonetheless.

Immature understandings 6, 7, and 8 Four Ss (17%) abandon the belief that the phenomena are random after having had an experience of high prediction success, not attributing the high success to chance. Two of these Ss experienced high success while using the prediction strategy of guessing a regular pattern, and the other two while using the prediction strategy of using the mechanism start state; all four Ss chose to use the prediction strategy for the money round for which they had experienced high prediction success. Three of the four Ss predicted that they would get a similarly high prediction success (7 or 8 out of 12) for the money round. Verbalizations of two Ss (labeled W and Z) illustrating their understanding, including their doubt that the phenomena are random, are provided here.

Exptr: [After initial demonstration of the Marble Shaking Machine and guessing game] How well do you think you would do with this tray? In other words, how many correct guesses do you think you'd get out of 12?
W: Four.
Exptr: OK. You say four. And why do you say four?
W: I would think that's about the odds at this point. I don't really know enough to, at this point really to say what the distribution within the tray is, so I'll just stick with four.
Exptr: What do you mean by the distribution within the tray?

W: Well, I mean, I would, I mean I don't know too much about physics (laugh). But I would think that if, if we had a lot of yellow marbles maybe bunched up together or something, closer to the center, that might influence the results. … I, at first I was looking to see whether the ones closer to the center would, would be the ones to go through. But they were pretty well shuffled (laugh).

Exptr: [For money round] What way of guessing will you use if you use this tray?
W: Looking for a pattern.
Exptr: Any particular pattern?
W: By twos.

Exptr: [Later] Could you say that these trays here were random?
W: Not conclusively.
Exptr: And what do you mean by that?
W: (pause) I haven't been allowed to have, I don't know all the properties of at work. I mean, I haven't had, I don't have enough trials with any of the trays to say anything. Nor do I know if the marbles are of equal weight or equal anything (laugh). Or if the trays are the same.

Exptr: [After initial demonstration of the Marble Shaking Machine and guessing game] How many correct guesses do you think you'd get out of 12, playing this game?
Z: Probably around five.
Exptr: And why do you say five?
Z: Just 'cause it's total chance. And, you know, I don't, I don't know.
Exptr: So five is like the chance level? Is that?
Z: (pause) Yeah, I guess. You know, that's, I just, I don't think I'm gonna do too much better than that.

Exptr: [For money round] Is there any particular way that you're going to use to guess with the second one [tray]?
Z: The way that I used for the second one [tray], which is to look at the marbles and see how they're placed around the hole, and try to guess which one will fall through.

Exptr: [Later] So would you call these trays here random?
Z: (pause) Well, in the way that I'm looking at them, I'm hoping that they're not. Because, you know, I want to be able to try to predict which marbles are going to come through. But, they probably are. I'd say they probably are.
Exptr: You'd say they *probably* are?
Z: Well, I don't, you know (long pause). I don't know if, I mean, I'm hoping that my method or being able to look at the marbles is gonna help, you know, me to figure out which ones are going to come through, but I, you know, it just may be chance that I did well with that one.
Exptr: So do you, so, how many do you think you're gonna get right using that method this time? OK? Do you think you're going to get seven? [as Z had predicted just before]
Z: Yeah, I'll leave it there.
Exptr: Because you just, it's just that, you know, you just said that they are *probably* random. You know, and I, cause I can see the two viewpoints.
Z: Yeah.
Exptr: You know, they're either random or they're not. And I'm just wondering which you think.
Z: (pause) OK. I'd say they're not. Because I have a, you know, I should have a better chance of getting more right, if I use the method that I was gonna use, which is, you know, looking at them, trying to, you know, see which one's going to come through the hole.

These Ss initially consider the phenomena to be chance or random, and then are influenced by receiving high prediction success using a prediction strategy, and end up expressing doubt that the phenomena are random, with three of the four Ss ending up predicting recurrence of similar high success in prediction. Through their judgments, they fail to show use of knowledge that high success in prediction may occur by chance as a rare event for a random phenomenon. They may express uncertainty in their judgments regarding predictability and the nature of the phenomena, but judge so nonetheless.

4. IMPLICATIONS FOR TEACHING PROBABILITY AND STATISTICS

Experimental results support the conclusion that a mature understanding of random phenomena, reflecting concepts and principles of probability and statistics (as represented in the cognitive schema introduced herein), is present for some, but is not predominant within a college population. For most college students, then, the mature schema (with the delimited/basic knowledge), which integrates information about random phenomena into a coherent whole, supports learning and retaining the mathematics of probability and statistics, and supports applying the mathematics to reasoning about random phenomena in the real world, is not evident in their judgments, actions, and verbalizations.

The content of the mature schema includes knowledge of calculating the probability and the expected frequency of an outcome, which is knowledge traditionally included in probability and statistics curricula. The schema also includes other knowledge such as that streaks of high prediction success may occur by chance as rare events, and that prediction strategies are irrelevant to prediction success, which are not traditionally a focus in teaching, but do have significant real-world application. The experimental results show that knowledge of calculating probability and expected frequency may be present without other elements of mature understanding, and that other elements of mature understanding may be present without the knowledge of calculation.

Building up students' cognitive schema for a mature understanding of random phenomena is proposed to be adopted as an explicit teaching objective to facilitate the learning, retaining, and applying of probability and statistics. Teaching would then include explicit presentation of the schema for mature understanding to facilitate seeing the integrated whole and the interrelationships among characteristics, and would include extended experiences interacting with common random phenomena that demonstrate the characteristics and establish them firmly in mind, to then be readily available to apply to real-world phenomena. The experimental results illustrate the diversity of immature understandings to be transformed to mature understanding.

Teaching probability and statistics supported with examples from common random phenomena and games of chance has the additional advantage that, in this historically original domain of application of probability and statistics, there is general consensus on how the mathematics of probability and statistics applies. For common random phenomena, the need is avoided for specialized domain-specific knowledge to ensure appropriate application of probability and statistics, and risk of misapplying probability and statistics may be reduced compared to other domains of application (Kuzmak, 2015).

5. CONCLUSIONS

Using historical perspective and educational context, the case was made that the development of the understanding of probability and statistics is dependent upon building a "mature" understanding of common random phenomena. A cognitive schema representing a mature understanding of common random phenomena was introduced, which integrates information about random phenomena into a coherent whole, including related mathematics (such as outcome probability, expected frequency, and variation). Experimental results of a study of 24 college students, who interacted with random phenomena involving the mixture of colored marbles, show that some students (17%) provided verbalizations that were consistent with the schema for mature understanding of random phenomena, while the majority (83%) provided verbalizations showing a diversity of immature understandings. The immature understandings observed included the belief that particular prediction strategies could provide an advantage in achieving prediction success for random phenomena (e.g., the gambler's fallacy), as well as the failure to use knowledge that high success in prediction may occur by chance as a rare event for a random phenomenon. Some students (25%) showed a lack of knowledge of calculating outcome probability and expected frequency; however, results showed that without this knowledge, other elements of mature understanding may be present. Also, calculation knowledge may be present without other elements of mature understanding. Building up students' cognitive schema for a mature understanding of random phenomena is proposed to be adopted as an explicit teaching objective to facilitate the learning, retaining, and applying of probability and statistics.

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