PROBABILITY MODELING AND THINKING: WHAT CAN WE LEARN FROM PRACTICE?

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ABSTRACT

Because new learning technologies are enabling students to build and explore probability models, we believe that there is a need to determine the big enduring ideas that underpin probabilistic thinking and modeling. By uncovering the elements of the thinking modes of expert users of probability models we aim to provide a base for the setting of new and more relevant goals for probability education in the 21st century. We interviewed seven practitioners, whose professional lives are centered on probability modeling over a diverse range of fields including the development of probability theory. A thematic analysis approach produced four frameworks: (1) probability modeling approaches; (2) probabilistic thinking approaches to a problem; (3) a probability modeling cycle; and (4) core building blocks for probabilistic thinking and modeling. The main finding was that seeing structure *and* applying structure *were important aspects of probability modeling. The implications of our findings for probability education are discussed.*

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1. INTRODUCTION

Within statistics education literature the characterization, definition, and explication of statistical thinking has been investigated and discussed (Chance, 2002; Ben-Zvi & Garfield, 2004; Wild & Pfannkuch, 1999). Similar discussions on characterizing probabilistic thinking are surprisingly limited (see Chernoff & Sriraman, 2014) as the main focus and concern in the literature has been on the ways that one can think about probability and on probability conceptions and misconceptions (e.g., Borovcnik & Kapadia, 2014; Savard, 2014). The way probability is viewed and conceived, whether from a classical, frequentist or Bayesian perspective, has been presented as fundamental to how one engages in probabilistic thinking. In this paper, however, we approach probability from a modeling perspective with the aim of characterizing some generic ways of thinking probabilistically.

The rationale for such an approach is in response to a recent paradigm shift in education towards considering that probability modeling may be better for enculturating students into probabilistic ways of thinking (Pratt, 2011). The current approach to teaching probability, particularly at the introductory level, draws on the tradition of classical mathematical probability, which may not be accompanied by a substantial understanding of the chance phenomena that mathematics can be used to describe (Moore, 1997). Often teaching approaches degenerate to a list of formulas and routine applications. Borovcnik (2011, p. 81) observed that, “probability is signified by a peculiar kind of *thinking*, which is not preserved in its mathematical conception.” Such a mathematical approach has resulted in many students unable to gain access to probabilistic ideas (e.g., Greer & Mukhopadhyay, 2005; Jones, 2005). Numerous education researchers and statisticians (e.g., Chaput, Girard, & Henry, 2011; Greer & Mukhopadhyay, 2005; Konold & Kazak, 2008) have emphasized that probability is about modeling real world systems in order to understand and make predictions about that system.

In this paper we aim to characterize probabilistic thinking and modeling through examining the thinking and practice of people who develop and use probability models. By probing the thinking of these experts we intend to glean ideas, principles, essential concepts and thought structures for the purposes of enhancing the teaching and learning of probability at the introductory level and for informing the end goals of probability education.

2. LITERATURE REVIEW

The thinking in probability modeling is not easy to describe, as the thinking seems to reside at the interface between intuitions and mathematical theory (Borovcnik, 2006). Borovcnik described probabilistic thinking as being a structure of thinking that is characterized by scenarios that allow one to explore reality:

> Concepts and models allow us to ‘see’ reality in a specific way – this acts as feedback also on our thinking, it structures our thinking insofar as it anchors analogies and figurative ideas and archetype models in our approach to reality. (p. 490)

When presented with a real-world situation one searches for simple analogies that have similar characteristic features to the situation, and thinks about model assumptions, while attempting to conceive reality in terms of random generating mechanisms. The idea of structuring reality surfaces in the work of Cobb (2007) when he was reflecting on teaching statistical thinking and the difficulty of transferring "thinking learned in one applied context to other contexts" (p. 338). Furthermore, he stated with respect to statistical models, "every instance of modeling is an instance of abstraction-as-process" (p. 339). His statement about statistics could be equally applied to probabilistic thinking. To him transfer and abstraction-as-process were major teaching goals, which he described with a tapestry metaphor. The structural threads of the tapestry provide the abstract concepts, which in the probability world could be concepts such as sample space and distribution, while the complementary threads provide the context, meaning and story. Through many examples in a range of contexts Cobb believed that students had to painstakingly build "an abstract understanding of what to look for" (p.339) and from these applied contexts also build intuition. To him "abstraction-as-process is our intellectual callisthenic" (p. 339) or a cornerstone on which to build the type of thinking that will endure despite changes in content and technology.

Our premise is that statistical thinking and probabilistic thinking are two sides of the same coin in the sense that practitioners faced with a problem will draw on either one or both perspectives and methods. In an attempt to separate the two approaches Biehler (1994, p. 2) stated:

> Probabilists seek to understand the world by constructing probability models, whereas EDA [Exploratory Data Analysis] people try to understand the world by analyzing data. Although both partly need the methods and concepts of the other, a tension exists. The tension is not new. The history of probability and statistics has seen different kinds of relations and roles of probability in data analysis, and vice versa.

The two approaches often reside within one person. Borovcnik (2006) explains there exists a mingling of probabilistic and statistical thinking with one or the other part dominating at different stages of attacking a problem. For example, when exploring data there will be a seeking of connections but also awareness that an observed pattern may be a chance artefact (Biehler, 1994; Wild & Pfannkuch, 1999). Explaining and describing the variation in the data by causal and other explanatory factors will be sought in tandem with constructing a model to make the pattern of variation more understandable and usefully predictable. The focus of interest may be on the exploration of the data set where use of contextual knowledge and statistical knowledge is integrated in an attempt to learn new information about the real-world situation but there will also be interest in producing probability models to generate distributions that are similar to the distributions of the data in an attempt to understand the behavior of and to predict future outcomes of the real world system (Pfannkuch & Ziedins, 2014).

The interaction and interconnection of statistical and probabilistic knowledge and thinking present a problem for novices who need to determine when to use one or the other or both. Biehler (1994) was also concerned that students immersed solely in the world of exploratory data analysis (EDA) would not cognitively integrate ideas from the probability-modeling world. Pratt (2011) echoed those concerns and noted how the statistics curriculum responded to opportunities afforded by EDA, enquiry-based problem solving and new technology yet the probability curriculum remained stationary "isolated in its strange world of coins, spinners, and dice as tools for demonstrating in a rough and ready way the *existence* of theoretical probability" (p. 892). He believed that a modeling approach to probability could "facilitate the re-connecting of probability to statistics" (p. 892) and provide a means for students to engage in a relevant world where real-world events could be simulated using computer-based simulations in much the same way that students engage in the virtual world of computer games. Moreover, such an approach could assist students to conceptualize the world non-deterministically since long-term learning experiences such as learning about long-run chance behavior through simulation and model building (Moore, 1997) and reflections upon probabilistic situations are necessary to overcome the many misconceptions associated with probability (e.g., Fischbein, 1975; Kahneman, 2011).

Recently, within some academic practice communities, there was a realization of the need to connect probability and statistics (e.g., Gimenez et al., 2014). There is also a belief that statistics and probability will not exist in isolation for much longer. In fact when identifying core changes in the ecology field Gimenez et al. (p. 2) stated “Statistical ecology has precisely solidified over the last decade as a discipline that moves away from describing patterns [in data] towards modeling the ecological processes that generate these patterns.”

Within mathematics education research modeling also seems to be gaining increased emphasis (e.g., Lesh, Hamilton, & Kaput, 2007). The purpose of creating models, according to Schwartz (2007), is indirectly to access unobtainable information about a system in order to explore and understand “the underlying mechanisms that govern the relationships among the entities” (p. 162). The information sought about the system from the model may be predictions, manipulations or explanations of outcomes (Thompson & Yoon, 2007). Models can be thought of as conceptual systems that are developed to make sense of situations and experiences while mathematical modeling includes quantifying, systematizing, and mathematizing (Lesh, Yoon, & Zawojewski, 2007) in a process that “involves translating between mathematics and reality in both directions” for which “mathematical ideas as well as real-world knowledge are necessary” (Blum, 2011, p. 17).

Amongst education researchers there is general agreement that the paradigm for modeling is an iterative cycle (e.g., Blum, 2011; Pfannkuch & Ziedins, 2014). The cycle, from a simplified perspective, starts with the real world system that is mathematized or mapped onto a model world system in which descriptions, predictions, explanations and “what if…” scenarios can be invoked for interrogation of the real phenomena. Finally the model solutions are validated against the real world situation and if necessary the model is revised and the cycle is repeated. For teaching and learning purposes, according to Doerr and Pratt (2008), there are two approaches to modeling: *building models* where the activity of the student is to build the model itself; and using pre-built models for the purpose of *exploring the behaviour of models*, in which students can investigate the consequences of actions and conditions such as varying input parameters and observing the resultant outputs. The latter type of modeling is aimed at developing students’ probability concepts. Henry Pollack (2007), a pioneer and proponent of using modeling in mathematics education, is in no doubt that modeling is a unifying force within mathematics and statistics and that modeling must be part of every learner’s experience from the primary to the tertiary level.

Spandaw (2011), however, found that such a perspective was not universally held. In his research to learn about mathematical modeling in tertiary education, he interviewed 12 research mathematicians, scientists, and engineers. To his surprise there was a persistent belief that modeling should be preceded by teaching basic mathematics and context knowledge despite the fact that “most students only managed to pass the theoretical exams after seeing applications of the theory in the computer modeling course” (p. 683). Also “most of the interviewees did not realize the possibilities to use modeling to *develop* mathematical concepts, not to mention meta-cognitive skills” (p. 686). They felt mathematical modeling should be restricted at the undergraduate introductory level to a few standard models akin to stating that probability teaching should be limited to classics such as the Poisson and Binomial Distributions. These interviewees' perspectives seemed to be centred on and limited to modeling as an application task (Stillman, 2012). An application task involves the student deciding what probability knowledge that they have already learnt and what pre-built probability model (e.g., Poisson) they should apply to a real-life type problem, whereas a probability-modeling task involves having to *build* a model.

Such a perspective does not take into account new technologies for learning that can now make previously inaccessible concepts and practices visible and accessible. For example, innovative ways to introduce middle school students to building probability models using specifically designed software, *TinkerPlots* (Konold & Miller, 2005), are being explored (e.g., Konold & Kazak, 2008). Some introductory university courses in the United States (e.g., Garfield, delMas, & Zieffler, 2012) are also using *TinkerPlots* to build probability models to introduce students to statistical inference. Furthermore, research using *TinkerPlots* has demonstrated that young students and introductory students can indeed build models for various problem scenarios to the extent that their cognitive infrastructure can change towards a probability-modeling type of thinking (e.g., Garfield et al., 2012).

When designing tasks for students using *TinkerPlots*, Konold and Kazak (2008) described four main conceptions that should be developed when building models: "model fit, distribution, signal-noise and the Law of Large Numbers" (p. 3). Developing students' probability conceptions through building models using *TinkerPlots* or through exploring the behaviour of pre-built models (e.g., Pratt, 2005; Prodromou, 2014) is a recent development. Traditionally probability education research has focussed on ascertaining students' probabilistic reasoning based on their interaction with mathematical probability tasks that a student might experience in a school or introductory curriculum. Hence in reviews of the literature Batenero and Sanchez (2005) and Savard (2014) identified research conducted on conceptions and misconceptions associated with combinatorics, sample space, sample size effect, randomness, conditional probability and independence, random variables and distributions, contingency tables, and simulations using coin, dice, and urn-type problems. Many of the misconceptions such as the equiprobability bias (Lecoutre, 1992), the outcome approach (Konold, 1989), and the heuristics of anchoring, representativeness, and availability (Kahneman, 2011) are well documented and have received considerable research attention in the educational and psychology fields. Batanero and Sanchez did mention, however, an area of research that seems to have received very little attention and that is the difficulty students have in identifying "the same mathematical structure in different probabilistic problems" (p. 249). Konold and Kazak also noticed how different problem contexts led students' thinking astray and believed that students must experience many contexts over many years in order to develop the ability to recognise how to link prior experiences to a new problem situation.

Both Batenero and Sanchez (2005) and Savard (2014) specifically indicated that their reviews of the literature were on probabilistic thinking. Although agreeing that all the concepts identified are part of probabilistic thinking we believe that an attempt should be made to characterize probabilistic thinking at a meta-level in order to gain a "big picture view" of the type of thinking that should be promoted in education. A review of the literature suggests there is limited research on developing students' reasoning in the actual field of probability at the undergraduate introductory level and little discussion on the type of probabilistic thinking that should be promoted within the arena of probability modeling.

3. RESEARCH QUESTIONS

As part of a larger project to enhance introductory probability students' thinking and reasoning using technologies for exploring the behavior of models, we began with the aim of uncovering and characterizing probability modeling and thinking through learning about the perspectives, thinking, and practice of practitioners. Hence our research questions for this paper are:

1. How do practitioners think when they undertake probability modeling?
2. How do practitioners characterize and perceive probability modeling?
3. What are the main building blocks that underpin probability modeling and thinking?

4. METHODOLOGY

The research method we used to uncover practitioners' perspectives on probability is thematic analysis, which Braun and Clarke (2006, p. 78) argue is a method in its own right and "should be seen as a foundational method for qualitative analysis." There are similarities to grounded theory in that we strived to change rich, thick descriptions from interviews into a structured way of representing the content of the interviews, which is parsimonious and in a diagrammatic form (Roth, 2005). However, unlike grounded theory we are not aiming to generate a fully formed theory grounded in the data, rather we are aiming to capture the essence of probabilistic thinking and modeling from the data in some frameworks.

4.1. PARTICIPANTS AND PROCEDURE

Participants in this study were seven practitioners ranging in age from their mid-twenties to late fifties who were selected according to their willingness to participate and in order to represent a variety of professional fields. See Table 1 for a description of the fields represented. All participants worked in collaborative or consulting roles involving interactions with other researchers or clients who did not necessarily have a background in probability. Some of the participants also worked in education as academic teachers of probability.

Table 1. Professional fields represented by interviewees

Interviewee	Field
Imogen	Queueing, networks
Rose	Ecology
Kristian	Probabilist
Margaret	Commercial
Gregor	Hydroelectricity
Alex	Agriculture
Thea	Operations management

Depending on the availability of the interviewee, individual interviews either took place in two one-hour blocks, or in one two-hour session. An interview protocol was provided to participants in advance of their interview and all interviews were carried out by the first two authors. Interviews were structured around two main areas, the first encompassing the interviewees' perspectives of probability together with the projects in which they were involved and problematic areas for their clients, and the second being their views on the essential concepts of probability for their students, employees, or themselves as students. The interview protocol took the form of a conversation and included the following starter questions: To you, what is probability? What is probability modeling? What is probabilistic reasoning? What are the concepts that you find yourself explaining to your clients so that they can understand the work you are doing? Tell us about two projects you have been involved in, and how you went about solving the problems. What do you believe are the essential concepts that students need to understand probability? Why? The interviews were audio- and video-taped and transcribed. It should be noted that client confidentiality was a concern to the interviewees and therefore they could only talk in generalities about projects in which they had been involved. Hence our reporting of their modeling projects needed to be circumspect. All participants were provided with the opportunity of reviewing this paper to check the authors' interpretations of their comments agreed with their intended meaning. Ethics permission for this study was granted by the University of Auckland Human Participants Ethics Committee.

4.2. DATA ANALYSIS

An inductive thematic qualitative data analysis was conducted on the interviews (Braun & Clarke, 2006). Thematic analysis is used for "identifying, analyzing and reporting patterns (themes) within data" (p. 79) and can be either inductive or theoretical in nature. The inductive approach means that the analysis is driven by the commentary provided in the interviews, resulting in the data informing the themes emerging, whereas the theoretical approach is more analyst-driven, with the purpose of shaping the data into a predetermined framework. The aim of this thematic qualitative analysis was to provide a detailed and authentic account of the interviews through the development and interpretation of common themes. It was undertaken by the first two authors and involved six phases:

1. Familiarization with data
2. Generation of initial codes
3. Identification of themes
4. Review of themes
5. Defining themes
6. Reporting

Phase 1, requiring complete immersion in the interview transcripts in order to gain insight into the data, was undertaken independently. The second phase involved highlighting interesting features or segments of the data in the transcripts. Initial codes were used to identify preliminary salient features or segments that might provide insight into the reasoning and thinking processes of the practitioners and their practice. Each feature was linked to the relevant data extracts. Hence each interview was mapped under four aspects (probability, modeling, thinking, and essential concepts), one aspect per A4 sheet of paper, through transferring the coding of the features initially noticed with key words and reference to the extract in the transcript. We then took one aspect at a time, for example modeling, and considered potential themes occurring across the seven interviews and the connection between these themes. Gradually these themes were formed into a map using post-it-notes for each aspect with main themes and connected subthemes being identified, merged, discarded, and modified. Thus the initial codes were considered throughout the third phase, contributing to the formation of potential themes, which were subsequently reviewed in the fourth phase in order to check that the data underlying each theme formed a coherent and meaningful summary. These latter three phases were undertaken through a series of meetings between the first two authors until consensus on the final themes was reached. At this stage a re-immersion in the data was undertaken to ensure that any previously un-coded data contributing to the overall picture was included in the analysis.

At the end of phase four, a check was made with the view that data "within themes should cohere together meaningfully, while there should be clear and identifiable distinctions between themes" (Braun & Clark, 2006, p. 91). Thematic diagrammatic maps were then created, allowing the identification of main themes and sub-themes, and the exploration of any interconnecting relationships. The remaining two phases involved final refinements to the themes and a presentation of the resulting themes to the project team. Feedback on the resulting themes and relationships was sought from the project team, which included people working in probability modeling fields, and led to further refinements to the thematic maps. Rather than being distinct entities, the phases were fluid, which allowed continual checks on the development of themes, to see where the relationships between themes and subthemes existed, and to make further enhancements to the thematic maps. These thematic maps were further refined with assistance from the project team into four frameworks to summarize the major themes at a meta-level.

The fifth author undertook an independent thematic analysis on the transcript data with the assistance of the software package NVivo 10 (QSR International, 2012). Welsh (2002) offers a metaphor that likens a qualitative research project to a rich tapestry with the software, i.e., NVivo, being "the loom that facilitates the knitting together of the tapestry, but the loom cannot determine the final picture on the tapestry" (p. 4). This image perfectly captures the role of NVivo in this particular research project. In a similar process untaken by the first two authors, the fifth author conducted an analysis of the data using NVivo. NVivo minimizes the time involved in the more laborious organizational tasks such as the coding and connecting of features and themes, thereby allowing a full immersion in the data in order to identify and interpret the emerging themes, and to consider the connections among the themes. This independent thematic analysis by the fifth author confirmed the main themes identified by the initial thematic analysis.

Further analysis and modifications continued after the interviewees read, commented on, and reacted to a draft of this paper. One issue raised was around the use of the term probabilist and the differing practices of theorists and modelers of real-world systems. Consequently only the theorist was named as a probabilist and another diagram, based on a thematic map created from the data but not used in the draft paper, was formulated to explain the different approaches to probability modeling depending on the role of the practitioner in the probability world.

5. FINDINGS

In this section we present the main themes that emerged from in-depth analyses of the interviews through the use of frameworks, which show how the identified themes or nodes were interlinked. The discussion on and the interpretation of the findings with regard to the themes and their interconnections are interspersed with quotes from the interviews chosen not only to illustrate the theme but also to provide evidence for the existence of the theme. Attention was paid to giving voice to all of the seven interviewees so that one perspective did not dominate the discussion of the findings.

Section 5.1 discusses how approaches to thinking and modeling depend on the role of the interviewee within the probability world. Sections 5.2 to 5.3 attempt to answer our three research questions: How do practitioners think when they undertake probability modeling? How do practitioners characterize and perceive probability modeling? What are the main building blocks that underpin probability modeling and thinking?

5.1. THINKING, MODELING, AND ROLES IN THE PROBABILITY WORLD

The type of probabilistic thinking and modeling used by the interviewees depended on their current role in the probability world (see Figure 1). Each of the interviewees played a particular role in the world of probability. The pyramid shape indicates the number of people who fulfil these roles with the modelers of real-world systems being more in abundance than the probability theorists or probabilists. The real-world modelers not only draw on the probabilists' work accumulated over many years but also provide the impetus to resolve particular theoretical issues that can arise through modeling real-world problems. Hence their work is inextricably interconnected.

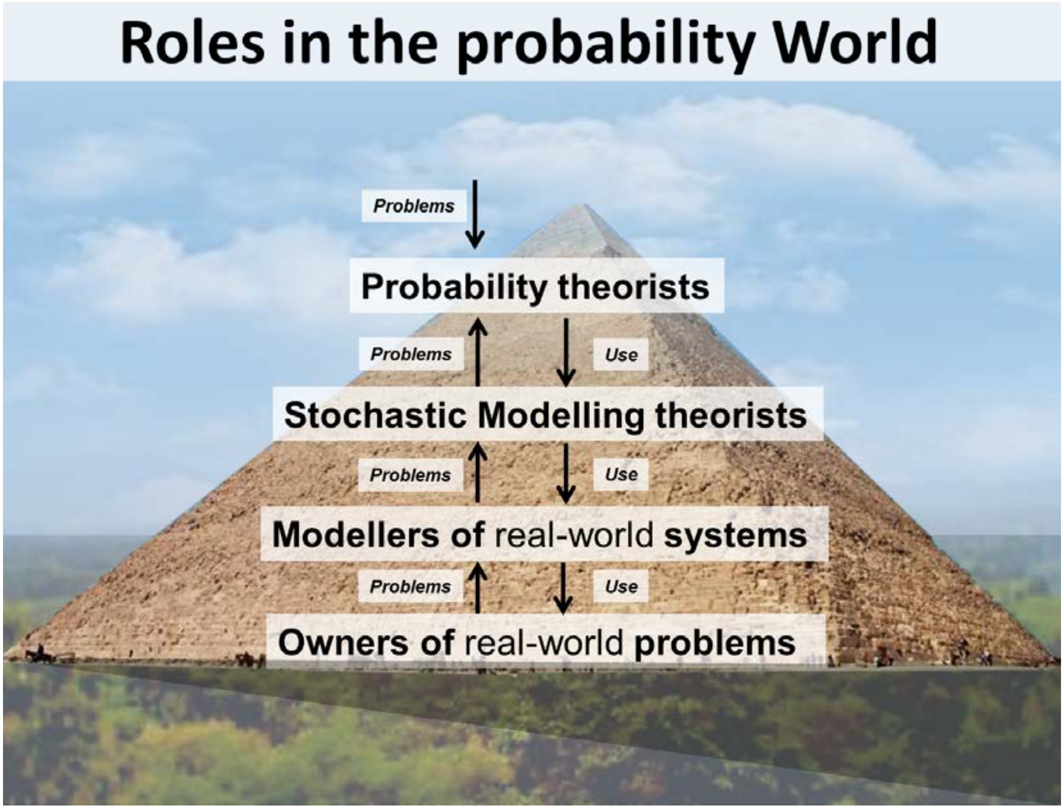

Figure 1. Roles in the probability world

From our interviews we ascertained that the probabilist, Kristian, was mainly immersed in the world of coins, spinners, dice, and random walks. Problems he was interested in could be described as mathematical problem situations. His thinking approach is to turn a verbal description of a problem into a purely mathematical problem with easy objects that he understands such as drawing on elements of coin-dice scenarios. "[I change] the words into mathematics, remove ambiguities and define all quantities to determine precisely what I mean and then it becomes a matter of a mathematics solution." That is, his main interest is in varying modeling assumptions and determining the implications that can be deduced mathematically from them.

At the other end of the spectrum were the modelers of real-world systems, Alex, Margaret and Thea. Alex, for example, recognized that dairy farmers in New Zealand were finding it difficult to make a decision about stocking rates when faced with possible drought.

> There is the uncertainty of the rainfall, some uncertainty about what you should do. Should I decrease stock early, should I buy in feed early, should I send some cows to the works, or dry off some of my cows early? [Farmers] think its complex and say I don't know whether it's going to rain, I'm just going to hang in there and then the last thing, oh, I've got no feed left I'm going to have to send some cows to the works and then the price for those culled cows is zero so they really make a big loss.

His thinking approach was to determine the variables contributing to being able to make a decision under uncertainty, to build a model using simulations to allow the farmer to input current information into the model he developed such as rainfall, stock level, grass level, and other parameters and then "the model says what optimum action to take such as drying off five cows." The components incorporated into the model were based on his probability scenario knowledge as he used an algorithm with four continuous state variables and a Markov chain determining the climate. That is, Alex was involved in constructing models and the model he built could be used by the farmer for prediction and decision-making purposes. By combining existing models, which can be underpinned by theory or empiricism, he built the model for the problem situation.

The other interviewees, Imogen, Rose, and Gregor, were also modelers of real-world systems but some of their work led them into the world of stochastic modeling theory to varying degrees. Imogen described how she was working on a problem derived from a real-world situation on traffic congestion and decisions in real time. To understand the problem her thinking was in terms of exploring idealized simplified models not driven by data. One of the questions was: "What is the effect of individuals having information about traffic congestion in 'real time' and making a decision based on that information rather than information based on average behaviour over many days and at the same time of day?" An individual has to decide what route to take based on real time data but also knows that other individuals have access to, and will be making decisions based on, the same information. Through the realization that game theory scenarios underpinned individual behaviour in these simplified models, Imogen was able to contribute new theoretical results to stochastic modeling. We would place such work under theoretical probability modeling as she was proving theorems mathematically and numerically exploring the behaviour of the models constructed.

Although the interviewees had different roles in the probability world their thinking on modeling was similar. They believed that extracting models such as the Poisson model from word-type problems and then mathematically obtaining the answers was an example of probability modeling, as was building models using computer-based tools. Hence in Figure 2 we propose a simplified way of explaining their perspective on modeling. Within the mathematical realm mathematical equations are the building blocks with which to construct models, whereas in the computer realm computer analogues from physical analogies (e.g., the spinners, urns, and distributions in *TinkerPlots)* are the component parts used to construct models. Once the models are constructed in both these worlds the behavior of the models can be explored, for example, to test the consequences of different parameter input values or for the purposes of what if… scenarios to predict the outcomes with certain inputs, which would not be possible to do in the real world system. The built model can be used by the model builder to explore possibilities such as Gregor did with his wind farm model with the consequence of discovering a better and unknown predictor for wind in the Manawatu region. An end-user such as a client (e.g., the farmers' example given by Alex) can also use a built model to see the consequences and predictions of changing system settings on the output of a model.

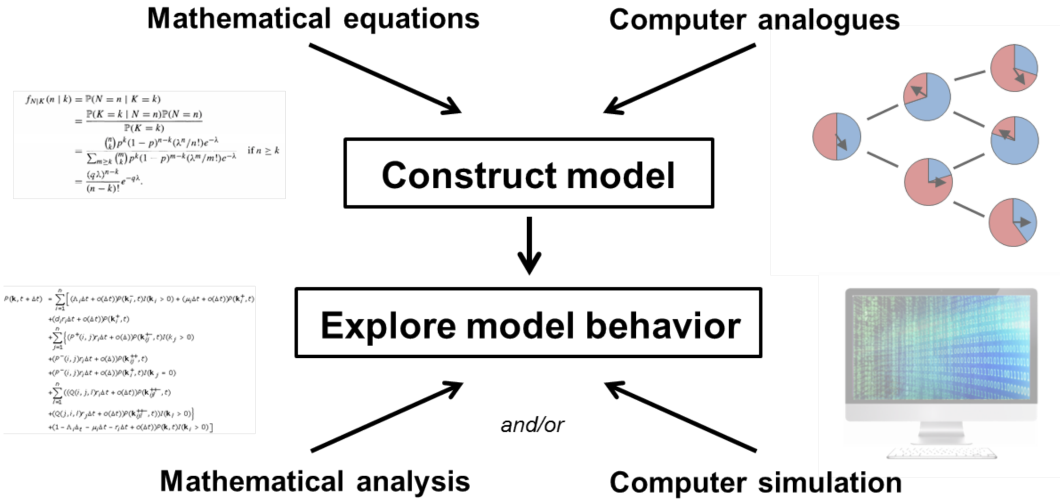

Figure 2. Probability modeling approaches framework

5.2. UNDERSTANDING THE PROBLEM SITUATION

In this section we highlight the thinking that is present when the interviewees are at the stage of understanding the problem situation. We demonstrate, however, that this thinking is also continually used throughout the probability modeling process (see Section 5.3 for more detail). Figure 3 shows, through the thickness of the arrows, contextual knowledge dominating at the beginning of the thinking process with probability knowledge to the fore as mapping to probability elements progresses towards constructing a model. All the elements of Figure 3 are interconnected, often occurring simultaneously within the thinkers as they attempt to understand how the real world system operates. To elaborate on each of the elements we give examples of how these interviewees described their thinking. Often in these examples there are a multitude of ideas but to enable coherence in their stories we focus on one aspect to demonstrate the elements in operation.

All the interviewees spoke of looking at the problem situation and *building a mental picture of the system.* Apart from the probabilist who drew on probability knowledge only, they immersed themselves in *the contextual background* in order to understand the dynamics and nuances of the problem system, while at the same time bringing their probability knowledge of scenarios and prior experiences to generate ways the system might potentially operate. To Margaret the contextual background was so important that she wrote it down in preparation for the interview, having been given the interview protocol beforehand:

> I wrote down really understanding your data in its context and its implications, thinking of the data you've got and what factors can influence that, just to get that intuitive understanding of how your variables relate to each other and how you think that's going to affect what you're interested in and is there anything else you would use. It helps you also explain it better and do that kind of sniff test at the end of "does this seem reasonable?"

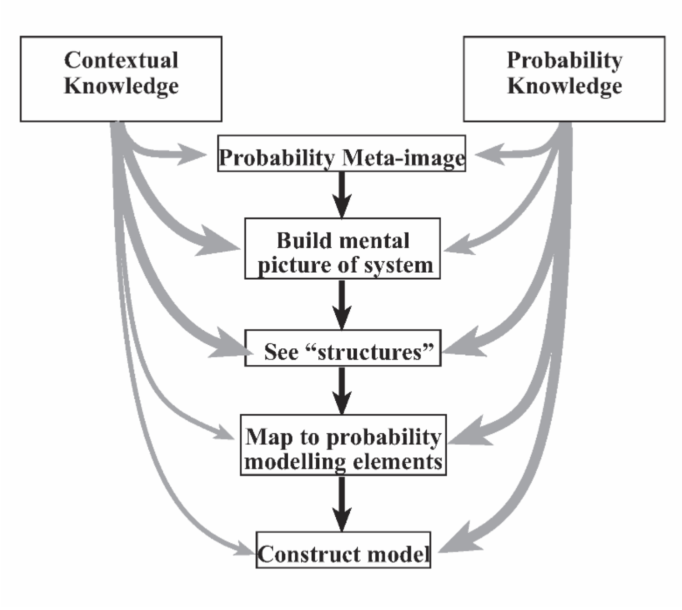

Figure 3. Framework for probabilistic thinking approaches to a problem

Such contextual background was also essential for the modeling process, for communicating results to clients and for determining whether the results made sense in terms of the real problem situation. Rose described a problem situation involving sea-lice type animals and circadian and lunar clocks. Her detailed knowledge about the behavior of these animals and how these clocks operated was essential for thinking of factors that might cause the animals to behave in certain ways both individually and collectively in the natural environment and in the laboratory environment. Through her considering potential animal behavior she set up three hypotheses to test in order to account for all the possibilities that could be occurring. Contextual knowledge was essential for working out how to model the situation, what simulations should be conducted and for the interpretation the data.

From the interviewees’ thinking about *building a mental picture of the system* we hypothesize that they seem to have *two probability meta-images* in their heads, a static probability image and a stochastic process image, which they drew on to *structure the system* and *map to potential probability elements* (Figure 3). We define a meta-image as a spatial picture, either internal or external, on which to structure thinking about a system and its inter-relationships. Margaret described how her primary visual image when faced with a marketing problem was a population or universal sample space which, using her contextual knowledge of the situation, she partitioned into segments, that is groups (e.g., by gender and age) formed so that behavior within each segment (e.g., reactions to a product, an advertisement) tends to be similar while the behavior between segments tends to be quite different. Each segment has a sample space with a conditional probability. “I think of different groups [sample spaces] and buckets [probabilities] and how they balance out. It’s all about balancing chance … it’s kind of optimizing uptake or optimizing responses.” While Margaret was explaining her thinking she vividly gestured this meta-image several times, which we describe as a *static probability image*.

The *stochastic process image* in its simplest form would be that of a queue, which Imogen used as an example of her thinking when in a very long queue to see Monet’s garden. “I went up to the head of the queue and actually did a calculation of how long roughly it would take us to get to the head of the queue based on the number of people that were ahead of us and how long it was taking to serve them.” She described how her perspective on such a situation was based on *building a mental picture of the system* through *seeing the structure* in something, awareness of questions one might ask about the system, awareness that things are random and that one might have a model for what was going to happen and considering data about the system. Similarly Gregor showed us his stochastic process diagram for hydroelectricity constructed in order to forecast electricity use and water conservation. His diagram partitioned the hydroelectricity system into a linked sequence of processes with feedback loops. His thinking focussed on randomness in the system (e.g., water inflow) passing information between parts of the system (e.g., forecast water level and expected cost) in an interplay between current information and observations and forecasting a week/month ahead. These meta-images seemed to help the interviewees' thinking as they structured or partitioned the problem situation into parts they could model, analyse, and link together.

Seeing "structures" within the real world system, or in the case of the probabilist seeing structure within the problem, was paramount and very important in the interviewees' thinking, which is built on the *probability knowledge and theory* they have learned and experienced over many years, both mathematically and empirically. Exposure to many scenarios such as disparate contexts where the underlying structure has Poisson properties, Markov chain processes, or even an unknown model enables thinking capable of structuring reality (cf. Cobb, 2007). Thea discussed some research she had conducted in manufacturing that drew on her knowledge of Poisson and Markov processes:

> Most of my research has uncertainty in it in one form or another. [I] was looking at manufacturing systems with set-up times, made to order, so you have uncertain arrivals. You also have uncertainty in the production times and in the switch-over times and I looked at just performance analysis, so basically sort of queuing analysis of how do these systems perform in terms of waiting times, etc. … at least 50 per cent of my work is Poisson arrivals, which is not a terrible assumption in a lot of cases.

We contend that her ability to see such structure in her problem situations was built on her experiences with Poisson-type process scenarios over many years. Similarly, the probabilist Kristian talked about using his knowledge of coin-dice scenarios in order to abstract the structure in a problem through defining events, random variables, conditional sample spaces, etc.

Sometimes there is no theory or empiricism with which to connect data. Thea described a problem where she did not feel "there was a good model of leave without being seen behavior" in relation to people leaving hospital emergency departments without being seen. In her thinking she could identify the probability modeling elements, the parts of the situation where randomness and conditioning should be considered such as the number of people in the waiting room, and had a mental model of the system but was unable to construct a satisfactory model of the system, so more empirical work was needed.

Rose described how it was necessary to experience both theory-driven probability and data-driven probability as they mutually reinforced each other.

> I would have to say that the data illuminate the theory perhaps more than the other way around, [the] modeling process being the way that you really get to understand it. Because I don't think I was a statistician really when I came out of my PhD. I was basically still a mathematician. I thought there were right and wrong answers and that symbols explained everything. And what's changed in the last 15 years is that I've used a whole lot of data and real life problems. And that I think has made me into a

> statistician, which I wasn't before. The theory only sort of takes you part way. I think with statistics there's a level of intuition or something else beyond where the symbols are acting. Like you've got theory and you've got data or practice and you need a little bit of this to start with and then you find there's a big surge of data sort of explaining the theory and then you get a better understanding of the theory so you can see more connection with more data, they are mutually reinforcing each other.

She went on to say that theory was necessary because "we have very poor intuition" with regards to chance and the theory "convinces me of the answer that I am giving to people." Hence the thinking about the structures present in the real world system is built on thinking about how data about the system can be connected to probability knowledge and theory (empirical and mathematical) learning experiences.

As the interviewees were thinking about seeing structures with the problem situation they were continually *mapping to probability modeling elements* such as sample space, assumptions, and known distributions in preparation for *construction of the model*.

5.3. THE PROBABILITY MODELING CYCLE

When Gregor was asked, "To you, what is probability modeling?" he replied that modeling is "like painting a picture. You know that no matter how good the picture is, it may never be confused with a photo. It will always look like a picture, but some pictures are better than others." In this section we focus on construction of the model and discuss what the interviewees considered to be probability modeling and how they think when they are undertaking probability modeling. We examine the various components of the probability modeling process, and how these components are linked to one another. A framework for the probability modeling process is proposed in Figure 4 with arrows demonstrating the pathways through this process.

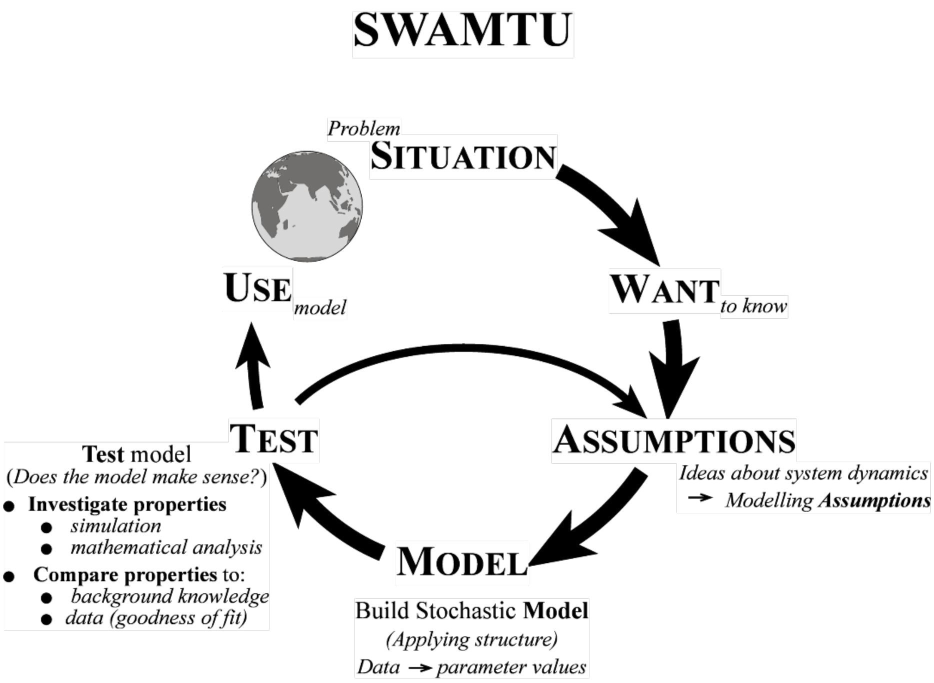

Figure 4. Framework for probability modeling cycle

As an illustration of the complete framework provided in Figure 4, consider the *problem situation* where there is interest in the effective management of a threatened animal species (based on a project described by Rose). Accurate and reliable monitoring of the population size is essential so that appropriate conservation strategies can be implemented. Researchers may *want to know* a variety of things such as the current population size, survival rates, the impact of a variety of threats on survival, etc. The usual approach to such a problem would be to use capture-recapture methods to estimate a variety of parameters of interest. Initially, fieldwork would provide capture-recapture data. Experience or judgment would inform the underlying *assumptions* of the system such as all animals having the same probability of being captured and recaptured, and tags placed on the animals not being lost between initial capture and possible recapture. These assumptions then generate the underlying *model* structure and the data are then used to estimate the model parameters. The *test* stage may involve a simulation to check on the adequacy of the model and if the model is found to be fit-for-purpose, it can be put to *use*. However, if the model fails the test stage because it does not adequately describe the data, or does not conform to background contextual knowledge of the situation, the *assumptions* need to be re-examined and the model re-specified. Thea also drew a similar framework in her interview to illustrate her overall approach to modeling.

Taking each node of the framework in turn, we now consider the underpinning components of the probability modeling process.

Problem/Situation Regardless of their role in the probability world (Figure 1), all of the interviewees talked of the start of the probability modeling process being a *problem* or *situation,* which may or may not reside in the real world. Modelers of real-world systems also acknowledged the need for contextual knowledge when setting out in the modeling process. While discussing a wind-farm forecasting problem, Gregor indicated that a lot of background contextual information was important for informing the modeling process. In order to begin the modeling process, he would need to study specialist wind farm literature and link this with his probability knowledge that would then feed in to the different stages of the modeling process (see Figure 3 and Section 5.2 for more on contextual and probability knowledge).

Want to know Once a problem or situation is identified, a *want to know* question provided the motivation for the probability modeling process. Imogen illustrated this idea using her Monet's garden story (see Section 5.2). On her visit to the garden, she observed that the queue was very long. Her *want to know* question was: How long would it take to get to the front of the queue? In a drought situation, the farmers mentioned by Alex (see Section 5.1) would also have a list of *want to know* questions such as: Should I decrease stock early? Should I buy in feed early? Should I send some cows to the works? Should I dry off some of my cows early? These questions feed indirectly to the probability modeling process because they define the type of decision or prediction the stochastic model should inform.

Assumptions The *assumptions* stage of the cycle encompasses ideas of system dynamics and *seeing structure* within the problem situation. A model is an approximation of reality and certain assumptions are required in deciding what type of approximation might be most appropriate. Rose spoke of "pick[ing] a law of chance that best describes what your situation requires." If a classic model is selected, for example, Poisson, then once a few assumptions are made "… suddenly you know all sorts of things about what to expect." Gregor made the point that for a given problem or situation there may be a range of existing (classical or pre-built/exploratory) models available for use but that the choice of the most appropriate model will be influenced by the ultimate goal of the modeling process. "And the interesting thing about that really is you know there are lots of models one could try to fit … but there are certain models that one wants to have because they lead into what one wants to do next.” In a situation such as this, links back to the probabilistic thinking employed in Figure 3 are evident. He is thinking of the purpose of the model, and noting that this will impact on the choice of structure he imposes. He noted “you have to know what’s really important so that you can come up with a model that reflects the important things and doesn’t waste a lot of effort on things that won’t matter in the end.” Again contextual knowledge, and its interplay with probability knowledge, plays an important role when seeing structure in a problem. It will often be an accumulation of experience, expert knowledge, and contextual knowledge that will inform modeling assumptions. As Rose points out, “So if your assumptions are incorrect then obviously what you end up with is not going to be very good.” As a simple example, based on Gregor’s description of modeling hydroelectricity, we would expect today’s weather conditions to be partially dependent on yesterday’s weather conditions, therefore independence of consecutive day’s weather conditions is probably not a good assumption for modeling the weather.

Build the stochastic model Again, all of the interviewees talked about *applying structure*, regardless of their role in the probability world (Figure 1). For probabilists such as Kristian, this meant abstracting from a verbal description of the problem by applying theoretical and mathematical constructs best suited to the underlying problem. Modelers of real-world systems noted that probability modeling involved imposing structure on a particular situation, and that their thinking was influenced by the type of problem they were trying to solve. In some cases, problems would be broken down into sub-problems, with a sub-model built for each compartmentalized section and then those sub-models would be linked together. Information may be passed from one section to another, with new information used to update the model (e.g., Gregor’s hydroelectricity model for forecasting demand). Building a model may also involve using data to estimate parameters. If a Poisson process were chosen as a suitable basis for some component of a model, data may be used to estimate the optimal value of the λ parameter. The data used for estimation purposes can also be historical. In one of Gregor’s examples he was interested in modeling water inflows to hydroelectric lakes. There are a number of climate-based scenarios to consider, each of which has a different influence on the water inflow. Part of the model building process involved attaching a probability to each of these scenarios and a natural approach to obtaining estimates for these probabilities would be “a historical record, this time last year, this time the year before, all the years we have records for.”

Test the model Several interviewees mentioned the role of *simulation* in the *test the model* stage of the probability modeling process. When a simulation is performed, the data generated by the model can be compared with the real data to see if the model produces data with similar properties and characteristics. As noted by Margaret, if simulations are performed under different conditions, it is possible to discover if the data generated by the model make sense. It may be that a model performs well under one set of conditions, but performs inadequately under different circumstances. If there is a disconnect between the real data and the data generated by the model, the model may not yet be fit-for-purpose and perhaps some of the underlying model assumptions need revision, or additional information or data are required. Rose explained:

> So if your assumptions are incorrect then obviously what you end up with is not going to be very good. But you would check this after having fitted the model, at a goodness-of-fit stage. So you would say, well, if this is the best fit that my model can provide, let’s compare the sort of data that would be generated by this best-fitting model with the real data. There will be certain attributes of the data that you care

> about – say, variance patterns, or autocorrelation – and you want to see if these attributes match approximately between the real data and the data generated by the model. And if they don't, that's when I say oh dear, bad model, start again kind of thing.

As indicated in the probability-modeling framework in Figure 4, an outcome of the *test the model* stage may result in revisiting the *assumptions* stage.

More formal tests of model adequacy include goodness-of-fit tests and data-driven validation tests. Rose explained that a goodness-of-fit test could be used to measure the appropriateness of using a Poisson process as part of the model, noting: "when a model doesn't fit, it's an opportunity to do something interesting." This would indicate a re-examination of *assumptions* and a re-consideration of the underlying structure. Furthermore, a mathematical analysis of the model may be used to derive particular properties of the system that are being modeled. Within a queueing-model context, Imogen spoke of a "mathematical analysis of the model", which involved examination of the implications of the model to understand what could be deduced about the behavior of the network and the behavior of individual queues. Such mathematical analyses may also be used to investigate what if… scenarios; for instance, what would the consequences be if more capacity were to be added to the system that is being modeled?

Is the model making appropriate decisions/predictions? Using the model to make decisions and predictions appeared to be the ultimate goal of most of the interviewees. Some of these decisions and predictions could remain in the model-world, with the model builder exploring a variety of possibilities by implementing what if… scenarios (see Gregor's wind farm model, Section 5.1). In addition to checking goodness-of-fit of the model, it is also important to consider the contextual background when deciding if the model is making appropriate decisions. For example, as explained by Gregor, the behavior of a stochastic model that has been built to describe water inflows to hydroelectric lakes will need to conform to both the observable real-world data, and with what matters to the hydroelectricity industry. Thus Margaret's notion of a "sniff-test" as a measure of the adequacy of the model applies to both the mathematical and the contextual fit of the model.

Use the model In the words of Imogen, "a model is only ever an approximation to the real world." When being used to make decisions or predictions, a model is not infallible. A decision based on the recommendation of a probability model will not always be the *best* one in the sense that the model is essentially built to optimize a particular situation. In Gregor's experience, a common issue is:

> The idea that if you make a decision on the basis of some stochastic model and then it might turn out to be, in hindsight, rather an important decision… A [electricity] trader might make a decision to buy power at a certain time. It might have been a good decision [based on the model] even though it doesn't turn out so well.

A probability model built to inform processes underlying a casino-operation does not result in the casino winning every game. As Gregor explains, "… you can still lose even though the odds are in your favor." However, the model will be built to ensure that the casino wins most of the games in the long run. In the context of storing inventory, Alex also alluded to the idea that a model is designed to optimize a particular situation. He noted that one stores items not only to meet demand, but also to cover fluctuations in the level of demand. A model is built to be optimal and Alex explained that:

> Even though it's not the best thing to do in any particular forecast for the future, it's optimal because of how much you should store and so on. You don't know what the future is, so probabilistic reasoning comes in to decide.

Thus using the model might not always result in success. As an approximation to reality, in many instances the model will work well, producing accurate predictions and informing good decisions. However, there may be times when the approximation to reality is inadequate, resulting in poor decisions.

5.4. WHAT ARE THE MAIN BUILDING BLOCKS THAT UNDERPIN PROBABILISTIC THINKING AND MODELING?

Seeing particular types of structure within a problem context is based on the development of an intuition or perspective of seeing the world through the lens of probability. Essential to the development of this intuition, which emerged from the interviews as necessary for probabilistic thinking and modeling (see Figure 5), are four main interconnected building blocks, namely: randomness, distribution, conditioning, and mathematics.

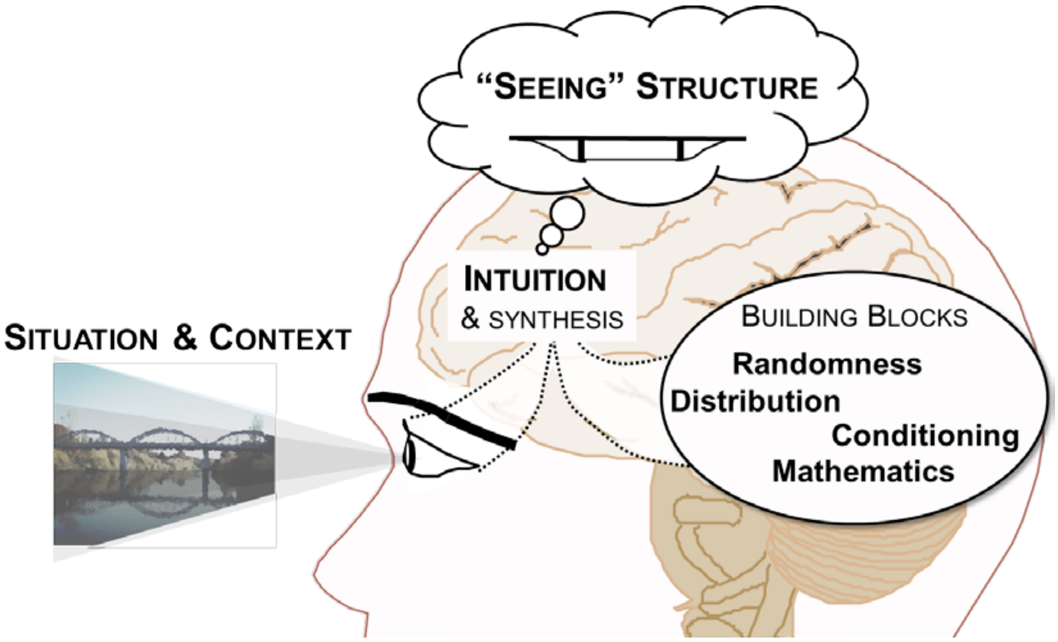

Figure 5. Framework of main building blocks for probabilistic thinking and modeling

Intuition, Context, Seeing structure Four of the interviewees specifically mentioned the word intuition in relation to being able to see the underlying structure within a context, where the context could be a real or mathematical problem situation.

> The most important thing in modeling is really understanding your data well … get that intuitive understanding of how your variables relate to each other. You choose a model and you see if that seems to be intuitively fitting the data that you've got. (Margaret)

> [There was this] Markov chain example where there was a lot of symmetry in the way the states were organized and so actually getting students first of all to see that that's the case and then what the implications of that are. You know developing students' intuition, understanding, and intuition for a problem. (Imogen)

Imogen believed that the way to develop students' intuitions to see underlying structures was to experience many diverse contexts and scenarios that they engaged in by themselves. Rose considered that being able to see structure was based in experiencing data-driven learning contexts as well as mathematical theory-driven learning contexts (see Section 5.2). When referring to learning contexts she stated that her mathematical approach to probability, which she experienced as a student, did not give her an "intuition" for data contexts. Apart from the probabilist Kristian, the other interviewees considered students needed both types of learning contexts to develop probability concepts.

Randomness Underpinning randomness are ideas about the effects of randomness, chance, variation, random variables, random processes, the Law of Large Numbers, uncertainty in situations, variance, and so forth. All of the interviewees mentioned randomness or uncertainty as the underlying motivation for using probability. The other elements in this list are ways of thinking about randomness and its consequences. In particular Thea stated that most of her research "has uncertainty in it in one form or another, uncertainty in arrival times, uncertainty in production times, uncertainty in switch-over times." Awareness of randomness or uncertainty led the interviewees to view a situation where they looked for it in parts of the system:

> For networks of various kinds you are thinking: where can individuals or items be, how do they move around from location to location, where are the sources of randomness? They may be in the way in which they move through the system, the time that they spend at various points in the system, when they enter, when they leave, all those things. (Imogen)

With regard to the effects of randomness Imogen described a situation she modeled of the number of patients in an Intensive Care Unit (ICU). She found it important to show her clients that her model mimicked the behavior of the ICU and actually reflected the reality that they knew. She showed them many simulations so that they could see the variation week by week and also to see that a simulation over many weeks gives average behavior but this is not what they will see from week to week. The distinction between average behavior over the long run and variation in the short term she further amplified by describing a queuing situation running at 70% capacity. People then think that they have 30% capacity available whereas if things are arriving very randomly "then the queue is going to get very, very long and you may not want that." She continued by saying that "it's just getting across the idea that mean values, which are almost a deterministic description of how the system is behaving, really aren't going to give you a very good handle on how busy it can get at times."

Rose believed variance was the key concept in ideas about randomness as "variance is the whole way that you are expressing the uncertainty, almost like the vehicle for randomness in a sense." Although the effects of randomness could be experienced in simulations, and the interviewees made extensive use of simulations in their work, there was still a need to capture and perceive "randomness" mathematically:

> [Chance is a] way of encapsulating the random unpredictable nature of the world into formal mathematical laws. So basically even chance, which should be something so unpredictable and unharnessable, even chance has to follow formal mathematical laws. There's chance that you can expect and there's chance that you can't expect in any particular situation and probability is all about understanding what is reasonable chance, how does chance behave in a particular situation. (Rose)

The ability to understand, capture and model randomness in a system, both mathematically and empirically, is a fundamental building block for probabilistic thinking and modeling. Randomness and how it behaves in a particular situation gives rise to learning about certain types of distributions and their properties, which then provide the underlying structure for modeling many diverse scenarios.

Distribution The importance of thinking about the context of the situation and the type of distribution that could be expected was an integral part of learning about probability modeling. Thea mentioned "the flaw of averages" in relation to distributional thinking for the type of models that could be used to generate distributions. She described a situation where she gave her students a macro from the material of Sam Savage, Stanford University, in which they were given a spinner and asked,

> to do a histogram of the pointer spin outcomes. And you get about half the class doing some sort of normal curve and you go look there's no weight that pulls the pointer down to 0.5, it's not like there's a weight on the bottom of it. It's equally likely. So therefore we have a uniform [distribution]. (Thea)

Distributional thinking in the form of recognising from a word problem or real world problem the underlying structure and assumptions that would reasonably describe a situation was necessary for probability modeling. As Rose explained there were often a few classic choices such as "Poisson for count data unless it is over-dispersed, when you might use a negative binomial; or for most scatter situations you might use the Normal distribution, or Gamma or log-Normal if it is right skewed." Imogen and Kristian talked about connecting real physical situations to an idealised base context such as coin, dice, urn, gambling, and random walk scenarios. That is, recognizing the simple problem in the embedded context. Kristian pointed out that a uniform distribution underlies drawing balls randomly out of an urn or selecting a person at random. For both theoretical and empirical modeling, building up a library of distributional contexts for these classical situations was considered important for laying foundations for seeing, determining, and applying structure to problem situations.

Gregor, however, said: "Binomial and Poisson are all nice and concrete, but of course most of the distributions one has to think about don't have a name. They are just distributions." He then demonstrated how he got students to appreciate a stochastic process where you "construct such a process so that certain key features of it match what the real world does. And then it's the data generating mechanism you are trying to come up with." For the data generating mechanism he showed us a scatterplot of some actual data; then using a combination of the Normal and uniform distributions he generated a similar-looking scatterplot.

> I think [it is important] to see what's going on with some of this stuff that we've talked about more abstractly in the past. So then you get some picture that sort of looks vaguely like pictures you get of real data when one thing is dependent on another. The whole painting a picture thing, when you try to put together a bunch of things that depend on each other and have some distributions in a way that looks right, looks like the thing you are trying to model.

Thea also got her students to generate from distributions a plot that was similar to a plot of data she gave them in order for them to understand the importance of distributional assumptions. Hence, appreciating that chance generating mechanisms underpin distributions and have the power to generate distributional patterns similar to real world data is an important part of probability distributional thinking, as is appreciating that classical distributions also provide the underlying structure for a diverse range of contexts.

Conditioning According to Rose conditioning is about understanding the inter-relationships within a system and how your knowledge about the situation "impacts on the sort of randomness you can expect to see." Conditioning is combined with a multitude of ideas such as sample space, independence, covariance, correlation, and expectation but to her and Margaret partitioning or the Partition Theorem is at the core. When presented with a problem situation both think of ways to partition or break the problem into parts that can be solved. All the interviewees described problem situations where they used conditional probability distributions and conditional expectation, and all seemed to be applying and thinking about partitioning. For example, Gregor described a situation where he was trying to predict whether there was going to be a peak in electricity demand on a particular night: "[We had] a conditional expectation, conditional on what we observed, what day of the week it is, what events we have already had this year, what the day's high temperature was, whether we had any of these events last night." Imogen also mentioned how she got students to explore a life expectancy applet (http://understandinguncertainty.org/survivalworldwide) in order to understand conditional expectation. Life expectancy is conditional on factors such as age, gender, and lifestyle factors, and the expectation changed when conditions or knowledge that one had was factored in. As Rose explained:

> Nobody really understands conditional probability until they have used it a lot. There's a whole kind of cascade of subtleties that follows once you start conditioning on things, you've got the interdependence and it changes. It's all about how your knowledge is changing the chances, it's changing your knowledge of chances, but it doesn't change actual outcomes.

Overall, conditioning underpinned by basic ideas such as sample space and independence was identified by the interviewees as one of the "big ideas" or building blocks for facilitation of seeing structure in problem situations.

Mathematics Probability can be viewed as a mathematical construct of reality. Mathematics is fundamental in probabilistic thinking and modeling ranging from number sense to laws and axioms. All the interviewees stated that fundamental mathematical knowledge, probability theory and notation, ability to manipulate symbols and to argue mathematically were essential for the introductory probability student. Imogen described how it was necessary for students to move to the symbolic formal mathematical world, as many problems were easier and much more manageable to solve within this world:

> You don't want a [probability] tree with a thousand branches, you're going to run into trouble. It's a good way of doing it. They understand it and that's excellent but you also want them to develop the mathematical part, the ability to write down the mathematics of it.

Alex gave a similar example of how students found probability trees easier to understand and so they used them but it limited their ability to engage with problems with many branches or conditions and the problem of generalization remained. He felt in his interview, as Imogen and Kristian did in their interviews, that students needed to wrestle and persevere with the mathematics as that enabled generalization.

Thea described a problem scenario of selling newspapers and predicting the right inventory level and how there was a very nice simple formula or mathematical model with a random variable D for random demand. Such a scenario led to a demand distribution where forecasting demand for newspapers depended on trade-offs such as whether the vendor would benefit more from selling the paper than having some left over. In this scenario the mathematics of the problem gave students a conceptual understanding or as she stated "an intuition" for modeling inventory type problems. That is, the mathematics illuminated the thinking behind the problem. Rose also described how mathematics lent conviction to and an understanding of the argument (see Section 5.2) in ways that empirical modeling could not. However, as Rose stated, mathematics on its own is insufficient for developing probabilistic thinking and the ability to see structure in problem situations. Empirical modeling allows one to experience and gain an intuition for randomness, distribution, and conditioning in a way that mathematics alone cannot.

The probability distribution models used, whether classical or generated from a combination of distributions, give rise to mathematical distributions; the gambling-type problems draw on mathematical understandings and are the base context for the rules of probability (e.g., Bayes' Theorem) that are presented and embedded in mathematical notation; what can be observed, empirically or theoretically, whether a distribution, life expectancy, or a probability of an event, can be expressed through mathematics. Hence, mathematics is a basic building block that underpins probabilistic thinking and modeling.

6. DISCUSSION

Our aim was to characterize probabilistic thinking from a modeling perspective and to characterize probability modeling through the perceptions and practices of some users of probability models. Working at the meta-level of thinking we are interested in those core-thinking processes that endure and are transferable despite new content knowledge and new practices arising from changes in technology. The main themes from the findings were *seeing structure* in situations and *applying structure,* which were variously described by the interviewees as recognizing structure or imposing structure on situations. Cobb (2007) referred to this type of thinking as abstraction-as-process whereas Borovcnik (2006, p. 490) referred to it as one where "concepts and models allow us to see reality in a specific way." We believe, in accordance with Borovcnik, that probabilistic thinking does reside at the interface between intuition and mathematics and show in Figure 5 how other main building blocks of conditioning, randomness, and distribution interconnect with mathematics. These main building blocks include specific conceptions and ideas about sample space, random variables, continuous and discrete distributions, and contingency tables. Engaging with these specific ideas, as Batanero and Sanchez (2005) and Savard (2014) suggest, does invoke probabilistic thinking but we contend at a meta-level of thinking our four main building blocks (Figure 5) operate together to enable the seeing of structure within a problem situation and to develop the intuition of seeing the world through the lens of probability.

Konold and Kazak (2008) identified model fit, distribution, signal-noise, and the Law of Large Numbers as main probability conceptions that should be developed. The former two conceptions we would classify in our framework (Figure 5) as part of the distribution main building block, whereas the latter two we would combine into the big idea of randomness. Because their elements were for middle school students they may not have considered mathematics and conditioning as appropriate for that age group. Nevertheless we would argue that even at that level these two building blocks should be a consideration for developing middle school students' probabilistic thinking and probability modeling. In alignment with Biehler's (1994) explanation of the culture of probabilistic thinking, our findings agree with his four points that users of probability models appreciate the effects of randomness, construct non-deterministic models, produce models using chance generating mechanisms to get "knowledge about the distribution in the population" (p. 7), and work with the model or probability distribution, a "deeper although not completely known reality, of which the data provide some imperfect image" (p. 7).

Contextual knowledge and probability knowledge worked in tandem as the interviewees built a mental picture of the system using their ability to see structure in the system and using a probability meta-image—a static probability image or a stochastic process image (Figure 3). Similarly, while applying structure in the modeling process, contextual knowledge and probability knowledge were interacting with one more dominant than the other as they moved through the stages of the SWAMTU (Situation, Want, Assumptions, Model, Test, Use) probability modeling cycle, which we proposed in Figure 4. Compared to the mathematics modeling cycle (e.g., Blum, 2011), SWAMTU is not only an augmentation of that cycle but also an illustration of the approach needed when building a stochastic model using chance-generating mechanisms. Similarly, SWAMTU is distinctive from the statistical empirical approach generally used such as the PPDAC (Problem, Plan, Data, Analysis, Conclusion) enquiry cycle (Wild & Pfannkuch, 1999). However, we think of the *AMT* part of SW*AMT*U as amplifying the nature of the analysis stage in the PPDAC cycle for a very important class of problems where the emphasis is on the role of randomness in the problem or the data generating mechanism; this is a characteristic that Biehler (1994) also noted occurring at the interface of probabilistic and statistical thinking. As a contrast to the PPDAC cycle we conjecture that the SWAMTU cycle may be helpful for illustrating and learning about the process of probability modeling for educational purposes as well as helping integration of the worlds of EDA and probability modeling that educators are seeking (e.g., Biehler; Pratt, 2011).

Based on the work of mathematics education in determining facets of mathematical modeling (e.g., Doerr & Pratt, 2008; Blum, 2011; Spandaw, 2011) we have defined probability modeling as being both in the theoretical and empirical worlds (Figure 1). In the theoretical world, modeling can be defined as envisaging model structures and using mathematical arguments to discover their implications. In the empirical world, modeling can be defined as applying structure to real-world problems through the building of a model that reflects the real situation and that is used to solve a problem. Both these worlds explore the behavior of models, which we define as the ability to change inputs such as parameters of models and explore the consequences of such actions on the outputs. Furthermore, we have shown how the role of a user of probability models determines his or her probabilistic thinking and modeling approach (Figure 1).

In characterizing probabilistic thinking and modeling we noted that both theory-driven and data-driven learning contexts were essential for development of such thinking. The mathematical world, with its symbolic structure, provides insights through the power of proven general results and rationales for arguments, whereas the empirical world, with its simulation nature, provides insights through visualizations of randomness and random processes and the ability to study more complex systems for which there are currently no theoretical results. Together these two worlds operate to produce the characteristics of the thinking of a user of probability models.

The proposed frameworks are based on the perspectives from one country and the interpretation of the researchers. We recognize that practitioners from other fields, countries, and cultures may espouse different viewpoints and that researcher bias and orientation may have led to particular interpretations and emphases. Another limitation is that the frameworks are based on the practitioners' perceptions on what they think they do when modeling situations rather than observations of their practice. Nevertheless, we believe that the frameworks that we have developed to characterize probabilistic thinking and modeling (Figures 2, 3, 4, & 5) have the potential to contribute to the knowledge base and to open up a wider perspective and discussion in probability education research.

7. IMPLICATIONS FOR TEACHING AND LEARNING AT THE INTRODUCTORY LEVEL

Currently many introductory probability courses reside mainly in the mathematical world with a focus on developing probability knowledge from a theoretical modeling perspective. Based on the roles of practitioners in the probability world (Figure 1) very few students will be engaged solely in the theoretical mathematical world. Our findings imply that better development of probabilistic thinking may occur if courses could strike a balance between using theoretical mathematical models, constructing empirical models, and exploring the behavior of models. Constructing empirical models for a problem situation will give students an opportunity to experience and to appreciate the power and purpose of modeling in understanding and predicting how a system works (Pfannkuch & Ziedins, 2014), whereas exploring the behavior of models has the potential to assist in the development of probability concepts (Spandaw, 2011). Meaningful simulations for the purpose of constructing models that mimic real systems will give students an appreciation of the capabilities of random generating mechanisms rather than the typical experience of doing simulations to show that empirical probabilities or probability distributions are similar to theoretical ones (Pratt, 2011). Such a balance between theory-driven and data-driven probability will also help in deepening conceptual understanding (cf. Spandaw) and enhance students' ability to move seamlessly between the theoretical mathematical world and empirical world.

Although the coin-dice problems are still essential as the base contexts from which to build probability knowledge, going beyond these contexts is also essential for learning to transfer to new problem situations. Seeing structure within problem situations draws upon a diverse range of learning experiences in multiple contexts. Seeing structure, abstracting structure, and applying structure to problem situations rests on a foundation of abstraction-as-process (Cobb, 2007) and using teaching strategies that deliberately enhance transferability to new situations. Possible learning strategies for seeing structure and transfer could be: the development of a list of critical questions to ask; paying attention to assisting students to transition from one representation to another, including the transition to the mathematical world of symbols and argumentation; using literacy strategies to unpack the meaning of a problem situation; and identifying transition blockages (Stillman, 2012) as students move through the modeling cycle from problem situation to solution.

Our findings suggest that probabilistic thinking and probability modeling are underpinned by learning experiences that encompass and develop an appreciation and awareness of randomness, conditioning, distribution and mathematics as the main building blocks within the probability world for imposing structure on problem situations. Probability provides a unique way of engaging with, thinking about, and interacting with a diverse range of real world situations where randomness and uncertainty are omnipresent. Learning to see the world through the lens of probability and to be enculturated in that world of thinking and practice should be the goal of education. With access to technology and the ability for teaching to incorporate new learning technologies (e.g., Konold & Kazak, 2008) it may be time to change the learning infrastructure within courses and programs and close the gap between education and practice.

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