GAME INVENTION AS MEANS TO STIMULATE PROBABILISTIC THINKING

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ABSTRACT

In this paper, we make a qualitative analysis of didactic experiments performed with five 6 to 10-year-old children and five primary school teachers, starting from a structured game with probabilistic elements. The fundamental idea is to stimulate probabilistic thinking not only by playing a card game with decision making in uncertain situations, but by inventing games modifying the initial game. These experiments are grounded on the importance of emotions for learning, the subjective probability approach, and researches on problem posing. We have found that this activity of inventing games has a significant impact on the development of probabilistic thinking in children and teachers; it reduces the anxiety and it could be used in teaching strategies to foster statistical and probability literacy. Some of its positive effects are the strengthening of creativity, self-efficacy, self-esteem, the ability to ask questions, and the enjoyment of learning.

Keywords: *Probabilistic thinking; Probability literacy; Problem posing; Game invention; Affective domain; Elementary education.*

1. INTRODUCTION

We believe it is very important to stimulate probabilistic thinking from an early age, and cultivate it throughout one's life. One reason for this is that uncertainty is omnipresent in everyday life. Just as Nikiforidou and Pange (2010) accentuate, "every event is characterised by a sort of estimation about its probable, possible, improbable, desirable or unlikely outcome" (p. 305). Another reason is that societies are moving in the direction of technological prevalence. Thus, in the new Big Data age, intensive information handling comes first in good decision making in daily life and in different professional fields (Mayer-Schönberger & Cukier, 2013). Therefore, as educators, we are committed to contribute to citizens' statistical and probability literacy so that they can behave in society in a more informed, analytical, and critical way. Furthermore, in the line of thinking of Nikiforidou (2018), our commitment begins with childhood since we should provide children with more and better motivating opportunities so that they can make sense out of the possible, random and impossible. As Yurovsky, Boyer, Smith, and Yu (2013) state, children develop their understanding of the world through causal and statistical reasoning based on experience in their environment, where the use of information helps them predict results and have an idea of what is likely and what is not.

In this context, games involve the learners in an active role of constructing mathematics. Thus, games are also intellectually helpful as the concepts to learn appear to be – in a natural

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way – meaningful. Games are inherent to the emergence of probability (Borovcnik & Kapadia, 2014). The concepts of probability and risk have been developed in order to improve one's situation in such games. Games, with their huge potential to generate different meaningful learnings, are a great basis to provide children with fun, challenging experiences with probabilistic situations, and thus instil probabilistic thinking in them, which we believe is even more stimulating when they are challenged to invent a game. A way of doing this is by asking children to invent their own games starting from modifying a previously structured game with probabilistic elements. This approach was developed in Malaspina and Malaspina (2017), which we now further study and expand, taking also in-service primary teachers in consideration. The approach in this study goes beyond using pre-specified games. The game presented first and played should be changed by the learners. This switch of perspectives demands a much more active role of the learners than only to play the game and find successful strategies.

It is positive that probabilities are part of the school curriculum but it is evidently not enough. To stimulate children's probabilistic thinking, it is fundamental to train teachers appropriately in this field, even more so considering that, just as Bryant and Nunes (2012) state, "despite the central importance of randomness and probability in our lives, it is clear that children, and many adults as well, often have great difficulty in thinking rationally about, and quantifying, probability" (p. 3). Thus, it is a great challenge to train teachers in probabilistic thinking, especially preschool and primary teachers, since there is little attention paid to this field in the initial training they get and because, when they get it, it is generally reduced to interpreting probability as a fraction considering every case in which an event can occur. Learning in informal contexts and games is not usually emphasised, even less so in psychological aspects tightly linked to probabilistic thinking, as Van Dooren (2014, p. 123) underlines:

> Besides mathematical challenges, probabilistic situations often also pose emotional challenges. Very often, probabilistic situations are not merely neutral to the problem solver, as the outcomes have a particular personal relevance and emotional, societal or material value: One strategy of playing a game may lead to a larger chance of success of winning a valuable prize, the implementation of a certain diagnostic screening may lead to the early detection of a rare disease, but with a risk of showing false positives (and sometimes even more false than true positives).

In the framework of the aforementioned reflections, this study poses the following research questions:

- Can playful situations in teaching stimulate children's probabilistic thinking and how can these games be used in teaching?
- How can teachers' mathematical and didactic knowledge of probabilistic thinking be improved and what role can playful situations assume in support?

Games are an essential part of the children life; in many of these games, the search for winning strategies is linked to probabilistic intuitions, which may be clarified in an accessible context. On the other hand, we believe that, similarly to the importance of problem posing in learning processes, beyond problem solving, it is essential not only to include playing guided games in learning processes especially if children are involved, but also focus on activities, in which they are challenged to invent their own games. We have developed this research with children and teachers in this perspective, starting from a game posed by Kamii (1995). We modified it in order to include probabilistic aspects, and we asked children and teachers – independently – to invent games by themselves from the one we proposed and played with them. This last phase – the invention of a game– is located in the framework of problem-posing researches. Inventing games that stimulate probabilistic thinking is also a way of posing problems and it merges playful activity with creativity, both of which are very typical of the age of the children that are involved.

2. THEORETICAL BACKGROUND

In this section, we summarize some approaches on probabilities and probabilistic thinking, as well as on processes of thinking in general, and we refer to the importance of probabilistic literacy. We also develop aspects of problem posing and game invention, explicitly stating their close relationship and highlighting their importance in the emotional aspects of mathematics learning in the context of stimulating the development of probabilistic thinking.

2.1 PROBABILISTIC THINKING

There are different ways of conceptualizing probability; according to Borovcnik (2016) and Nikiforidou (2018), there are three theoretical approaches that have received more recognition and are considered the most important ones to explain the nature of probability. The first one is the classical interpretation, in which probability is explained as a fraction of a total number of possibilities where events occur. The second theoretical approach is the frequentist, which is described by Batanero, Chernoff, Engel, Lee, and Sánchez (2016) as "the hypothetical number towards which the relative frequency tends when a random experiment is repeated infinitely many times" (p. 4). Finally, the third one is the theory of subjective probability, which can be interpreted as the outcome of a preference system that depends on a person's knowledge or experience and that is signified by biases and heuristics, which emerge from the interplay between intuitions partial knowledge. From this preference system, a degree of belief may be derived for specific statements (Borovcnik, 2016; Nikiforidou, 2018). In that sense, already Kahneman and Tversky (1972) stated that a probabilistic approach plays an essential role in our lives since "the decisions we make, the conclusions we reach, and the explanations we offer are usually based on our judgments of the likelihood of uncertain events" (p. 25). For this reason, as Fischbein and Schnarch (1997) mention, probability should not only be seen as the implementation of techniques and procedures to reach solutions but also as a specific way of thinking, in which intuitions are omnipresent.

Thinking processes are generally linked to intuitions; according to Kahneman (2011), there are two mind systems:

- System 1 operates fast, and automatically; it is intuitive and emotional.
- System 2 is slower, deliberative, and logical; it requires attention and focus.

When making decisions, these systems interact, and one way of interaction occurs with the intuitive suggestions System 1 gives to System 2 as well as the rational reactions of the latter. This general overview is coherent with what Bishop (1981) states regarding the two ways of thinking needed in mathematics: creative thinking, for which intuition is typical; and analytical thinking, for which logical reasoning is typical. These ways of thinking are different, but they complement each other.

Focusing more specifically on probabilistic thinking, it is becoming more and more important in society since, as Batanero and Chernoff (2018, p. v) state:

> To adequately function in society, citizens need to adapt their deterministic thinking and embrace chance and uncertainty in different settings. At the same time, they need to acquire strategies and ways of reasoning that help them in making appropriate decisions in everyday and professional situations where chance is present.

In this perspective, researches have been developed on statistical literacy and probability literacy. Gal (2005) poses a model for probability literacy, emphasising abilities that need to be developed in order to be able to interpret and critically analyse probabilistic information and random phenomena; likewise, Gal states that, based on psychological literature (Kahneman, Slovic, & Tversky, 1982), human opinion will always be affected by the context in which an information is given. Gal poses identifies five knowledge elements and some further dispositional elements as the basic building blocks of probability literacy.

Borovcnik (2016) maintains that mathematical competencies do neither play an obvious nor a substantial role in Gal's probability literacy model and he adds to the components of probability literacy "the ability to use relevant concepts and methods in everyday context and problems" (p. 1500). As Batanero and Chernoff (2018) point out, the importance of probabilistic literacy is illustrated by the fact that, in many countries, school authorities have recognised the need for teaching probability to everyone; for this reason, probability is being included more and more in schools and in teacher training. This inclusion of probability into the curricula is even more relevant as research shows that people widely use inadequate (non-probabilistic or probabilistic) strategies and have an over-confidence in their approach and the ensuing results. Studies indicate a very low level of probabilistic thinking (Van Dooren, 2014).

In the last decades, different disciplines and theoretical frameworks, such as the mathematical, cognitive, and educational approaches have focused on studying the development of probabilistic thinking (Nikiforidou & Pange, 2010). However, in order to have a historical perspective, it is important to remember that this topic has been studied for more than 60 years (Chernoff & Sriraman, 2014a). Jones and Thornton (2005) locate the researches on probabilistic thinking in three periods, which are described below.

Piagetian Period. Between 1950 and 1960; this period is signified by investigations in the field of psychology and cognitive development, especially by Piaget and Inhelder (1951/1975). Piaget and Inhelder mainly studied children, their development, and the structure of their probabilistic thinking. They consider three developmental stages of the idea of chance: pre-operational, concrete operational and formal operational; and they state that children in specific stages are prone to idiosyncratic intuitions. Batanero, Chernoff, Engel, Lee, & Sánchez (2016) mention that early research on probabilistic thinking has been initiated by investigations on intuition and learning difficulties.

Post-Piagetian Period. Between 1970 and 1980 approximately; this period is still characterised by the strong influence of Piaget's researches; yet, gradually the research paradigm switched to the development of people's probabilistic ideas, heuristics, and intuitions. Fischbein's (1975) approach is worth mentioning in this period since he provided a more general educational perspective of probabilistic thinking in children and considered intuition as a process of understanding probabilistic notions; he made a difference between primary intuitions (not related to formal education) and secondary intuitions (related to formal education). As he had a pronounced interest in probability, he transferred these ideas to probability learning. Also in the psychological field, Tversky and Kahneman (1974) made a meaningful contribution in this period by their studies, which focused on heuristics or strategies that people use to make probabilistic judgments (e.g., representativeness, availability and adjustment, and anchoring); they also analysed the prevalence of systematic errors resulting from biased thinking (Kahneman, Slovic, & Tversky, 1982). This period also marked the beginning of researches from the field of mathematics education and, above all, investigating the effect of teaching the subject on the development of probabilistic thinking in children; however, most researches had a classical approach on probability as a reference framework and only a few included a frequentist or subjective approach (Jones & Thornton, 2005).

Contemporary Research Period. Approaches and researches since the 1990s; this period is influenced by great curriculum reforms in mathematics education worldwide (in the US, e.g., National Council of Teachers of Mathematics [NCTM], 1989, 2000), in which training in probability was promoted at different levels of education. In parallel, there was a significant growth in research on probability teaching and learning. In this period, research was more oriented towards the needs of the curriculum reform and the actual teaching in the classroom (Jones & Thornton, 2005).

Finally, it is worth mentioning that researches on probabilistic thinking are on-going and get deeper by developing theories, models, and frameworks associated to intuition and learning difficulties. Given this situation, Chernoff and Sriraman (2014b) suggest to consider a new phase since 2010 called *Assimilation Period*, which may be characterised as "renaissance period for psychological research in mathematics education" (p. 723).

2.2 PROBLEM POSING AND GAME INVENTION

There are numerous studies and didactic experiments carried out on problem posing. Singer, Ellerton, and Cai (2015) provide a broad vision of the paradigm of researches done in this field. They refer to investigations that link problem posing to general mathematics training and the development of abilities, attitudes, and creativity. English and Watson (2015) use problem posing as a means to make a significant contribution to the development of children's statistical literacy. It is worth mentioning that different researchers (Kar, Özdemir, İpek, & Albayrak, 2010; Nicolaou & Philippou, 2007; Silver & Cai, 1996) have found a meaningful correlation between the performance in problem-posing and in problem-solving, both in students and prospective elementary teachers. Even more so, as Chang, Wu, Weng, and Sung (2012) mention, "efficacy beliefs in problem-posing could predict mathematical achievement fairly well" (p. 775).

On the other hand, there are many studies that accentuate the importance of factors such as emotions and motivation in learning in general (Pekrun, 2014). This is a research strand in the affective domain of mathematics education that has been growing a lot in the last decade due to its valuable contributions (Grootenboer & Marshman, 2016; Schukajlow, Rakoczy, & Pekrun, 2017; Xolocotzin, 2017). For example, Lyons and Beilock (2012) emphasise that anxiety – often generated while teaching and learning mathematics – is an emotional factor that influences learning strongly negatively. For this reason, games in teaching and learning have gained more attention in the general didactics as well as in the didactics of mathematics (Hassinger-Das, Zosh, Hirsh-Pasek, & Golinkoff, 2018). Games can induce valuable contributions at a motivational *and* a creative level (Cecchin, 2013). Providing children with playful occasions for the development of their mathematical thinking is in line with Tall (2013), who states, "as individuals take personal routes through their development of mathematical thinking, human emotions play a significant role in supporting or inhibiting progress" (p. 23). That is to say,

- games can generate positive emotions;
- there are many analogies between games and mathematics (Schoenfeld, 1992); and
- there are researches on valuable contributions from problem posing to processes in mathematics learning (Singer, Ellerton, & Cai, 2015).

Transferring these results and developments in the general mathematics didactics to teaching probability, we consider it as very important to explicitly link problem posing in mathematics to game *invention* to stimulate probabilistic thinking from early childhood on. Danniels and Pyle (2018) identify essentially two types of games that have been studied in relation to their benefits for learning: "free play, which is directed by the children themselves, and guided play, which is play that has some level of teacher guidance or involvement" (p. 1). We consider that these two types of games can complement and boost each other if participants are asked to invent a game inspired by a guided play, having them create their own rules.

In our problem-posing approach, we consider four basic elements of problems: *information, requirement, context*, and *mathematical environment* (Malaspina, 2017; Malaspina, Mallart, & Font, 2015). Similar elements can be found in structured games to stimulate mathematical thinking since it is essential

- to have information (rules, materials),
- to have a requirement (the objective of the game, what needs to be achieved in order to win the game),

- to link to a context (which can also be intra- or extra-mathematical), and
- to embed the game to the mathematical environment (made up of the global mathematical framework where mathematical concepts that intervene or may intervene are located to develop the game).

In our approach we also consider that posing a new problem is done by *variation* or *elaboration*. Variation is done by modifying one or more elements of a given problem. Elaboration is done by more thought on a given situation due to a specific requirement with mathematic or didactic emphasis. We can directly transfer this structure to game invention: A new game is invented by *variation* from a known one or by *elaboration* from a given situation or specific requirement. Most certainly, when considering inventing such games in teaching and learning contexts, the specific mathematical cognitive objectives need to be investigated in the framework of intuition development and mathematical thinking.

3. METHODOLOGY

In the framework of qualitative methodology, we performed a case study in Lima, Peru, with five children of age 6, 7, 8, 9 and 10 years (two girls and three boys) and with five in-service primary teachers (four women and one man) ranging from age 38 to 52 years with an average of 18 years in teaching service. Three of them had postgraduate studies and, in terms of their teaching experience, they covered all grades of primary education.

3.1 THE DIDACTIC EXPERIMENT

The game

Inspired by the game called *War*, which Kamii (1995) proposed for preschool children, we invented a card game so that we could investigate elements of children's probabilistic thinking and their decision making in a context characterised by uncertainty. We called this game "*Change?*" To play the game, 16 cards (Figure 1) are needed with values from 1 to 4 (the 1 is represented by the ace) in the usual four suits (spades, hearts, diamonds and clubs).

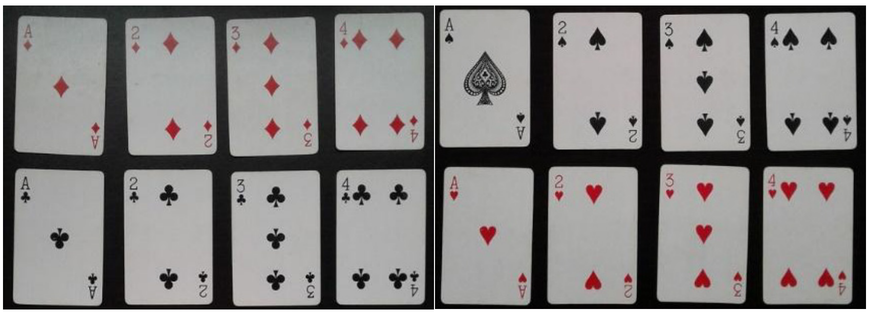

Figure 1. Full set of cards for the game "Change?"

The rules of "Change?" as a two-person game:

- At the beginning, the 16 cards are shuffled, piled in a stack, and lie face down on the table, so that none of the players can see the values on the cards.
- Each round, each player draws one card from the deck; no other player can see the value on the card.
- Each player shows the card; the player holding the highest-value card takes both cards. In case of a draw, each player takes one card.

- Before showing their cards, players can exchange their card for a different one from the stack if they wish to do that. The exchanged cards are removed from the game.
- The game continues until all cards initially placed on the table are gone. The winner is the player who holds more cards at the end.

It was investigated as a two-person game. However, it could well be played with more players. Obviously, the strategies would change and get much more complicated.

In this game, just as in mathematical problems, we can perceive the four basic elements: information (16 cards and the game rules), a requirement (gain as many cards as possible), a context (extra-mathematical, card game), and the mathematical environment (order relation between natural numbers and probabilities). In that sense, and similarly to the problem-posing approach we already described, we asked the participants (the children and the in-service primary teachers) to invent a new game based on this game.

Game sessions

We had individual didactic experiments with each child and a few weeks later with each teacher. In each case, we started playing the designed game with the child or the teacher. During the game, we observed the reactions of the participants and the strategies they adopted in order to win the game. Then, we asked them to invent a new game by introducing modifications to the present one; that is, in terms of problem posing, we asked them to invent a new game by variation of the game that was initially introduced to them. They could modify one or more elements of the game already described. We observed the reactions and strategies they adopted when playing the game they invented, and we finally discussed with them about their reasoning, reactions, and emotions in both games.

3.2 STAGES OF GAMING FOR THE ANALYSIS OF THE EXPERIMENT

Hereunder, we explain the stages we followed in the game sessions to generate the subsequent analysis:

Stage 1. Understanding the game and looking for winning strategies
The game was played twice or more in order to facilitate an understanding of the game and support the children and teachers in planning their own strategies for winning.

Stage 2. Inventing a new game and searching for winning strategies
We proposed to each participant separately to invent a new game based on the experience in playing "*Change*"; the new game should be similar to this game but they should invent their own rules. First they had to explain the rules of this new game then we proceeded to play it together.

Stage 3. Reflections on reactions and strategies used by the participants
After the games we had a discussion with each child using questions such as the following:

- Did you like the first game? What did you like best (or not) about the game?
- We asked you to invent a game. How did you feel in that moment?
- Did you like the game you invented? What did you like best (or not) about the game? How do you feel about the game you invented?
- Which game did you like best?
- What did you do to win the first game? If you had a card with a number 2 on it, did you change the card? If you had a card with a 3 on it, did you change the card? Why?
- What did you try to do to win the game you invented?

For the teachers, we used a questionnaire and after they finished it, we had an informal discussion with them about some of their answers. Some questions were:

- What strategies did you follow to try to win the game?
- What motivated you to make these modifications to the initial game?
- In comparison to the initial game, do you think the game you invented is
  ( ) less fun ( ) just as fun ( ) more fun
  Why?
- In comparison to the initial game, do you think the game you invented is
  ( ) less challenging ( ) just as challenging ( ) more challenging
  Why?
- Which of the situations was the most challenging for you to make a decision in the games played? Why?
- Which mathematical concepts do you think are involved in the games played?

4. RESULTS AND DISCUSSION

When exposing the participants to the initial game, we observed that both children and teachers quickly understood the rules of the game and were interested in playing it. During the game, we realised how System 1 and System 2 of thinking of Kahneman (2011) were reflected in the decisions the participants made to change their card or not before comparing it with the opponent's card, which is unknown to them. The immediate, intuitive decision to change the card when getting a 1 and not to change it when getting a 4 is located in System 1 and every participant applied these criteria. The decision to change a 3 or not – which evidently requires deeper thought and prompts to consider the probability of getting a 4 when changing the card – is located in thinking of System 2. Most of them preferred not to change the 3; however, the cases in which they decided to change this card (especially younger children) were when they had won with a 4 before. This may be explained in the framework of subjective probability since individuals base their decisions on past experiences and intuition (as Borovcnik, 2016, and Nikiforidou, 2018, mention); yet, biases also intervene in their decision making. It is essential to note that, as more rounds of the game were played, these participants no longer changed a card with a 3 on it.

In the conversations we had with the teachers, none of them explained the criteria for changing a card with a 3, neither did we find clear explanations for the underlying reasoning behind the way they played in the initial game or the one proposed by them when these games included probabilistic aspects. Some of them said, "The luckiest wins" or "I would never change a 3". These results are startling because of the low level of probabilistic thinking they reveal as school teachers play a key role in the foundation of probabilistic literality of the next generation. Yet, games and everyday problems are valuable inputs for developing probability literacy in the sense of Gal's (2005) and Borovcnik's (2016) approach. Thus, we identify here a need for teacher education to focus on such games and game invention as this would improve their probability literacy and boost their capacity to teach probability successfully.

In our didactic experiments with children and with teachers, game invention by variation has been more present, modifying the information (the number of cards or the rules) or the requirement (the objective of the game). Now we perform specific analyses, where we investigate Tables 1 and 2 as a reference summarising the modifications made to the initial game by the children and the teachers in inventing their own games. The first column in both tables shows the basic components of the game, whilst the second column includes the setting of the initial game, that is, of "*Change?*" The other columns include the modifications made by the children or by the teachers. We used the = sign to indicate when there were no modifications made to the initial game.

Table 1. Modifications made by the children to the initial game

Game components	Initial game	Boy (6)	Girl (7)	Boy (8)	Boy (9)	Girl (10)
Number of players	2	=	=	=	=	=
Number of cards in the game	16	24	24	32	40	32
Cards drawn per round	1	=	4	2	10→1	=
Maximum changes in each round	1	=	2	2	2	=
Criterion, which player takes the cards in a round	Highest-value card	=	Highest sum of cards	=	Lowest-value card	=
Winner	More cards at the end	=	=	=	=	=

All children understood it as a challenge to invent a new game inspired by the initial game. Some of their reactions were, “Oh, cool! I will do it!”, “I will invent a nicer game”. All of them managed to invent one and the analogic mathematical thinking from System 1 was evidently present. We have noticed that, the older the child, the less spontaneous the invention of the new game was, and that older children tended to locate their thinking more in System 2 than in System 1 right from the beginning. For example, the 9- and 10-year-old children took more time to invent a new game and the 10-year-old girl only changed the number of cards.

All children – without actually saying it – included probabilistic aspects in the games they invented and their game had a greater cognitive demand than the initial one. In general terms, the trend in the suggested changes to the game was to increase the total number of cards as well as the number of cards to draw or change in each round. In the case of the 7-year-old girl and the 9-year-old boy, there were changes in the criterion to decide, which player takes the cards in a round; yet, they kept the decision to a point in time when the situation was still uncertain. The 9-year-old child introduced the rule that the player who takes all the cards in a round is the one holding the card of lower value. The game invented by the 7-year-old girl is worth noting: she used 24 cards from 1 to 6; each player drew 4 cards per round and they could change two of them; after swapping cards, each player showed their 4 cards and the one with the highest sum won. The addition of four numbers, without paper and pencil, was challenging for the girl. However, she noticed very well that in some cases the addition was superfluous, when it was immediately clear that one player had very high numbers while the other one had very low ones. It is worth mentioning that we highlight this invention due to its originality in comparison to the other games invented; however, the girl focused more on arithmetic aspects rather than probabilistic ones. This seems logical, due to the fact that little attention is paid to probabilistic thinking in school and emphasis is given to aspects of calculus.

We noticed that teachers reacted less spontaneously than children, and they were more concerned or taken aback in the beginning when inventing a new game. Some of the reactions were the following: “Invent a game? What do you mean?” or “Invent a new game right now?” Then, after spending much more time than children and using pencil and paper, they managed to propose their invented game. It is worth mentioning that, in some cases, there was a tendency to propose a card game they already knew, even though it was not similar to the initial game proposed. We consider this is closely related to the fear of making mistakes in society and not seeing them as learning opportunities, particularly in the face of new situations. It is important to mention that this could also explain the loss of spontaneity, to which we referred when analyzing the games invented by older children. It seems like the spontaneity gets lost with age while at the same time the fear of committing errors increases.

From Table 2, we can state that Kahneman's (2011) System 2 prevailed, without a clear interaction with System 1, since they used more complex rules than the initial game; the rules were even difficult to understand and had greater emphasis on arithmetic than intuition. It is remarkable that four teachers included probabilistic aspects in their invented games. Three of them posed situations quite similar to those of the initial game and Teacher 3 proposed a more complex probabilistic situation by adding the colour of the cards as a variable.

Table 2. Modifications made by the teachers to the initial game

Game components	Initial game	Teacher 1	Teacher 2	Teacher 3	Teacher 4	Teacher 5
Number of players	2	=	=	=	=	=
Number of cards	16	32	20	32	40	20
Cards drawn per round	1	=	=	=	Not used	=
Maximum changes per round	1	=	1 Only the first to say "change" can change the card	=	Not used	=
Criterion, which player takes the cards in a round	Highest card	= *but* cannot take the cards if the opponent immediately says the difference of the cards on the table	Highest value between the card of a player and the sum of his opponent's two cards. Whoever decides to change his card has to give it to the other player and take a new one from the stack. The value of this card is compared with the sum of the two cards in the opponent's hand.	Black card regardless of its value. If the two cards shown have the same colour, the highest value	The first to notice that the card of the opponent is even	Lowest card
Winner	More cards	Sum of the values of the cards	=	Black cards with the highest sum	=	=

Teacher 4's proposal was not similar to the initial game and the emphasis lied actually on quickly recognising an even number in the opponent's card. At the beginning of this game, each player has half of the cards and they cannot swap them in each round. For this reason, "Not used" was included in the third and fourth rows of Table 2. Her proposal reveals that she did not identify the mathematical environment of the initial game, and this is consistent with her not mentioning probabilities when we asked her about the mathematical concepts she found related to the game. She only mentioned counting and comparing natural numbers.

In the interviews, we noticed that the teachers did not naturally link the games to probabilistic situations and did not recognise their potential to stimulate probabilistic intuitions in children. They saw only numeric or arithmetic aspects in them such as number comparison or operations with numbers. Once again, this reveals the low level or lack of appropriate use of probabilistic thinking in society just as Batanero and Chernoff (2018) or Van Dooren (2014) state it, and also the little emphasis laid on non-deterministic situations and concepts in teacher training.

Even though children – and teachers – find it fun to play structured games that pose a challenge, we have observed that it is also fun and, in many cases, even more motivating to play challenging games invented by the participants themselves (in the experiment that applies more to the children than to the teachers) by modifying rules from a structured game, or to play games that are new with their own new rules. Some of the children's reactions were, "Let's play again" or "I like playing my game more than yours". The emotions perceived and the challenge to know how to exploit them in this playful context to contribute to the development of probabilistic thinking remind us that in every creating action, intellectual and emotional factors are just as necessary, and that feelings and thoughts are the factors that move human creation (Vygotsky, 2004). Moreover, he states:

> One of the most important areas of child and educational psychology is the issue of creativity in children, the development of this creativity and its significance to the child's general development and maturation. We can identify creative processes in children at the very earliest ages, especially in their play. (p. 11)

It should be noted that, in every case, there was more emotional involvement and motivation after inventing the game, especially when playing it, even though in some cases they had strong difficulties to explain the rules of the invented game. A boy told us, "My game is nicer than yours, but I don't know how to explain it". This is a reminder of the emotions provoked by problem posing in mathematics, as well as the difficulties in writing a formulation without leading to ambiguity to the others. Likewise, in the case of children, we were able to perceive greater emotion and motivation to find winning strategies for the games they invented by themselves, although the invented games had a greater cognitive demand than the initial game. This was a great opportunity to learn more due to the greater challenge for their probabilistic thinking.

5. FINAL CONSIDERATIONS

The experiments carried out underline that game invention by modifying a given game is understood as a natural part of playful activities, strongly supported by intuition, emotion, and motivation. To play with rules set by the children themselves – modifying the rules given in a game – is an experience that contributes to their self-learning, stimulates their creativity, strengthens their self-esteem, improves their attitudes to statistics, and reduces their anxiety. In this line, Nikiforidou, Pange, and Chadjipadelis (2013) state, "play is a means to get children involved in problem solving situations and develop their thinking on mathematical ideas and procedures" (p. 349). Even more so, this is included in Ben-Zvi's (2018) perspective, who states that "today's students need to learn to work and think with data and chance from an early age, so they begin to prepare for the data-driven society in which they live" (p. vii).

The experiments performed with the teachers make us propose that, in the framework of didactic-mathematical competencies primary teachers should have, it is very important to focus on creative activities. A fundamental part of this is problem posing and game invention. This is even more important in the case of probabilities because, even though it is so relevant in children education and though it is so present in their games, its teaching in schools is minimised or developed with little emphasis on playful situations from a more classic or frequentist perspective just as Ben-Zvi (2018) states, “despite the recognised importance of developing young learners’ early statistical and probabilistic reasoning and conceptual understanding, the evidence base to support such a development is rare” (p. viii).

The aforementioned considerations and proposals are reinforced after observing the didactic experiments with children and teachers, since both groups pretty much had the same gaps in their mathematical training in regards to probabilities and basically made similar changes, showing a similar comprehension of what they were doing. The curriculum of primary education in Peru considers problem solving of data management and uncertainty as a competence; however, emphasis is really given to aspects of descriptive statistics, rather than to probabilistic thinking. On the other hand, in the initial and continuing primary teachers’ preparation, few hours are dedicated to mathematical contents, and probabilities are weakly considered among them or not considered at all.

It is worth mentioning that, in our didactic experiments in teacher training workshops, posing problems or inventing games with greater cognitive demand is quite frequent. We could find that this approach provides valuable opportunities for reflection at a didactic level (Malaspina, Torres, & Rubio, 2019). This research takes into account aspects developed on probabilistic thinking in the Post-Piagetian Period, emphasising its intuitive nature and the heuristics. Likewise, valuing the reflections on the Contemporary Research Period, this study intends to contribute providing an innovative proposal on probability teaching and learning. Finally, it breaks new ground to integrate and apply more and more psychological and emotional aspects through games and game invention in mathematics education in the framework of the Assimilation Period.

When focusing on children, especially those in their first years of elementary education, we consider that it is really important for statistics teachers and psychologists to meet the challenge to propose game invention as part of the strategies to stimulate probabilistic thinking, taking advantage of the researches on problem posing, which show its great potential at a cognitive and emotional level. Another interesting, innovative perspective is given by the contributions of design learning for game invention; an experiment within this approach was elaborated in Malaspina, Malaspina, and Malaspina (2018), where groups of students invented, designed, and prototyped games for primary-school children in order to stimulate the development of some types of mathematical thinking, one of them being probabilistic thinking.

We consider that this study, particularly the aforementioned children’s expressions, contribute to the statement that playful activities are an excellent means to stimulate probabilistic thinking and mathematics learning in a more attractive, fun way in informal contexts, where children show less mathematical anxiety. In this perspective, Hirsh-Pasek and Golinkoff (2008, p. 3) state that “when children play they are learning. Children who engage in play and playful learning do better in academic subjects than their peers who play less.” We suggest incorporating playful activities that include game invention for children since the preschool curriculum as well as in the teacher-training curricula, especially the curricula of primary teachers. In this sense, similarly to problem posing (Malaspina, Mallart, & Font, 2015), games can be invented by variation of given games or by elaboration from described or configured situations. We emphasise the importance of games with uncertainty components, with which the development of an intuitive comprehension of probabilities can be stimulated in children, primary teachers, and future statisticians, as the basis of a later formal study.

Certainly, this suggestion also involves a commitment from statisticians and statistics educators to propose attractive games with the ability to stimulate probabilistic intuition and to provide rich opportunities to invent new games. There are aspects that have not been addressed, such as cooperative games, games with more than two players, games with body movements, and the use of technological resources in games. All these variants of games provide us with opportunities to broaden and deepen the researches on the role of games and the invention of games to stimulate the development of probabilistic thinking at very early ages. This approach is aligned with that of Perry and Docket's (2007, p. 2), who state that "with social interaction providing support and the play context creating a situation where innovation, risk taking and creative problem solving can all be encouraged, young children learn a great deal about themselves, others, and the world in which they exist."

When thinking about the education of future generations, it is important to have Vygotsky's (2004) words in mind as a great general framework: "the main educational objective of teaching is guidance of school children's behavior so as to prepare them for the future; development and exercise of the imagination should be one of the main forces enlisted for the attainment of this goal" (p. 88). Unfortunately, in regards to educating future generations of statisticians, statistics educators have just recently started to make proposals to develop stochastic ideas in primary schools. We know how important it is to start at an early age and to revise conceptions in various stages to let the individual thinking specific to that area gradually develop and ripe.

While there are successful attempts for teaching elementary statistical considerations and methods (descriptive statistics, tables, diagrams, statistical figures representing location and spread), it seems much more difficult for probabilistic thinking and there are only a few studies with teaching experiments on probability such as Martignon and Krauss (2009), or Martignon and Hoffrage (2019). By the present study, we try to contribute to the aim of early education in probability. Our experiment with the game "Change?" corroborates that game invention has a high didactic potential: the game invention approach may be used to teach probability at primary-school level. A sound and flexible basis of probabilistic thinking may soften the impact of our raw and often not helpful primary intuitions with probabilistic concepts and thus pave the way for a probabilistically more educated next generation, which may also encourage more young persons to specialise in that field. On top of the pyramid of the probabilistically more experienced next generation, we may find a larger group of statistical experts whom we urgently need in the next future.

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