CRITICAL THINKING AS AN APPROACH TO COORDINATING PERSONAL, EXPERIMENTAL AND THEORETICAL NOTIONS OF CHANCE

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To foster secondary students' probabilistic thinking, the coordination of personal, experimental, and theoretical notions of chance is necessary but rather challenging. Whereas critical thinking is effective in terms of self-awareness and self-regulation of learning, it also has the potential as an approach to overcome the challenge of coordinating the three notions. This study aimed to propose a framework for task design and analysis based on an enactivist view of cognition and a reformulated learning model, Judgment-Enactment-Reflection integrated with critical thinking. Tasks were designed according to the framework and implemented in two Grade-9 classes. Results indicated that the personal, experimental, and theoretical notions of chance could be coordinated to develop the subjectivist, frequentist and classical perspectives of probability with critical thinking as an approach.

INTRODUCTION

For the concept of probability, researchers have emphasized three perspectives of probability, namely classical, frequentist, and subjectivist, and their coordination (Batanero et al., 2005). To clarify the three perspectives, Batanero et al. (2005) acknowledged the importance of "interpreting random situations in terms of probabilistic models" (p. 32). Random situations could be distinguished as either symmetric or asymmetric. Symmetric random situations assume "outcomes are equally likely if we have no reason to believe that one or another outcome is more likely to arise" (Borovcnik & Kapadia, 2014, p. 20). Based on this assumption, "there is not much room for acknowledging and discussing the mathematical meaning and values of relative frequencies of an event" (Park & Kim, 2023, p. 441). Thus, asymmetric random situations, in which equal likelihood for each outcome based on trials cannot be assumed, are recommended for helping students to coordinate the classical and frequentist perspectives of probability (Park & Kim, 2023). The subjectivist perspective of probability was not considered in Park and Kim's (2023) study, which designed a modeling activity to stimulate students' coordination of the classical and frequentist perspectives of probability.

When including the subjectivist perspective, symmetric random situations could be interpreted as asymmetric random situations and vice versa, according to a personal notion of chance. Then, an experimental notion, as the basis of the frequentist approach, is required for convincing oneself and others whether predictions or judgments for the random situations are optimal. Cognitive psychology has suggested that changing misconceptions is difficult as people are more likely to ignore, reject, or reinterpret anomalous data to maintain their prior beliefs or notions, rather than to reorganize their belief systems (Chinn & Brewer, 1993). Empirical studies have confirmed that students who are better at critical thinking are more likely to change their misconceptions (e.g., Jolley et al., 2022; Kowalski & Taylor, 2004). Critically evaluating quantitative information and thinking critically about uncertain situations are also part of probabilistic literacy (Gal, 2005). Thus, the teaching of critical thinking that supports probabilistic thinking and vice versa, appears to be a fruitful educational endeavor.

To foster secondary school students' probabilistic thinking, this study proposed a framework for designing tasks and analyzing students' learning while performing the tasks. First, we illustrated the three notions of chance and clarified their difference from three perspectives of probability. Next, the construct of critical thinking and its pedagogical function were elaborated. Then, to modify a prediction-observation-explanation learning model based on an enactivist view of cognition, a conceptual framework was proposed and was applied to the design of the tasks. Last, the tasks were implemented in two grade 9 classes, and students' learning while performing the tasks was analyzed to explicate students' probabilistic thinking.

THEORETICAL BACKGROUND

Probability has been considered as having classical, frequentist, and subjective perspectives (Batanero, 2020). The classical perspective is based on the assumption of equiprobability for all possible elementary outcomes (Borovcnik & Kapadia, 2014). This definition was criticized as the assumption of equiprobability of the outcomes may rely on subjective judgment. The assumption of

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equiprobability could be warranted by personal belief and mathematical convention (Chernoff & Zazkis, 2011). The former is referred to as a "personal notion," while the latter is referred to as a "theoretical notion". For an uncertain situation, a personal notion may be involved with or without a theoretical notion. For instance, although students have the same judgment that the outcome (1,3) for the experiment of rolling two dice is as likely as the outcome (2,2), they may have different reasons. One may be that they are respectively one of all equiprobable outcomes, and another may be that both are one possible outcome. The former student's personal notion is involved with a theoretical notion, but that of the latter student is not.

The shift from three perspectives of probability (subjectivist, frequentist, and classical) to three notions of chance (personal, experimental, and theoretical) is due to the rationale of this study adopting an enactivist perspective of experiential learning. This perspective assumes that "cognition depends on the kinds of experience that come from having a body with various sensorimotor capacities embedded in a biological, psychological, cultural context" (Fenwick, 2011, p. 47). That is, cognition co-emerges in complex systems where the learners, the tasks, and the social contexts bind together and interact with each other (Maturana & Varela, 1987), and learning is occasioned as "reflections of a structure without losing sight of the directness of our own experience" (Varela et al., 1991, p. 12). Accordingly, notions of chance are considered as an individual cognitive domain of meaningfulness initiated and evolved in complex systems, and are different from perspectives of probability which represent epistemic accounts of probability.

The importance of critical thinking to academic performance, creativity, work practice, and participation in public is widely accepted and supported by research. Paul and Elder (2005) viewed critical thinking as "a process by which the thinker improves the quality of his or her thinking by skillfully taking charge of the structures inherent in thinking and imposing intellectual standards upon them" (p. 1). This view implies critical thinking is related to metacognition, that is, cognition about cognition, and is required for self-awareness and self-regulation of misconceptions and errors (Borasi, 1994; Dellantonio & Pastore, 2021; Hartman, 2001). Accordingly, tasks for learning probability are suggested to facilitate students' critical thinking so they can have the opportunity to be aware of, regulate, and coordinate their notions of chance. According to Yang et al.'s (2023) study on assessing secondary school students' critical thinking of mathematical literacy, critical thinking involves the three key components of perceiving the possible invalidity of an argument, explaining for refutation, and evaluating the validity of inferences. The first component is related to being aware of others' misinterpretations, which are a kind of error. Then, we think about the design of tasks for eliciting students' multiple notions of chance which further facilitate students' critical thinking, and then the need to regulate and coordinate the personal, experimental and theoretical notions.

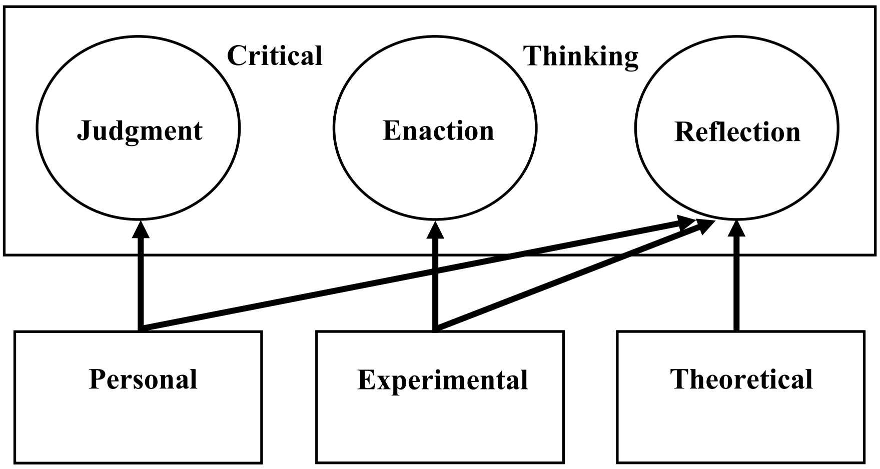

Figure 1. A Framework for Task Design and Analysis of Learning Probability

A Judgment-Enaction-Reflection learning model, adapted from the prediction-observation-explanation model (White & Gunstone,1992), involves critical thinking to challenge and co-construct students' notions of chance as a framework for task design and analysis. The prediction-observation-explanation was originally proposed to promote students' conceptual changes, and popularly applied in science education. According to this model, three types of tasks would be designed: predicting what will happen based on their own justifications, observing what really happens, and explaining the possible inconsistency or consistency between their prediction and observation.

The rationale of this model, however, is based on a constructivist rather than an enactivist view of cognition. The key difference between constructivist and enactivist views of cognition is that an intersubjective sphere is emphasized by enactivists, but disregarded by constructivists. Based on the intersubjective sphere, enactivists pay more attention to developing habits of mind through a freeform-trial-and-error interaction between learners (Simionescu-Panait, 2020). Thus, Prediction-Observation-Explanation is adapted as Judgment-Enaction-Reflection, which considers the intimate and reciprocal interaction between learners, tasks, and social contexts.

METHOD

The study adopted a design-based research method including three main phases: the conceptual, developmental, and experimental phases (Barab & Squire, 2004). In the conceptual phase, a framework was formulated to develop four main learning tasks, which is the second phase. The experimental phase consisted of a classroom teaching and formative evaluation. A framework was proposed in the Theoretical Background. The four tasks were designed accordingly by a team, and were implemented in two grade 9 classes by the same teacher. It took 90 minutes to implement all the tasks. The learning tasks and students' responses were analyzed in three phases: a Judgment-Enaction-Reflection learning model, critical thinking, and notions of chance. A short version of teaching videos can be accessed using the link of https://www.youtube.com/watch?v=dS7Hh7LFKtM.

RESULTS

Before presenting the analysis of the students' learning, Table 1 summarizes the characteristics of the four tasks. Eliciting, exploring, extending, and coordinating activities appeared between 0:01-1:12, 1:12-8:03, 8:03-14:37 and 14:37-17:05, respectively, in the video.

Table 1. Brief Description of the Four Tasks

	Eliciting	**Exploring**	**Extending**	**Coordinating**
Judgment	Making a decision.	Making a prediction.	Making an estimation.	Making an estimation and a decision.
Enaction	Picking one item from the bag they chose	Doing experiments.	Imagining different ways of experiments. (Thought experiments)	Reasoning in an imaginable situation based on a hypothetical and numerical premise.
Reflection	Expressing feelings	Contrasting the original prediction and the experimental results.	Explaining the difference between the experimental and theoretical notions for their predictions.	Reasoning in the same situation based on an add-on hypothesis.

The first task asked students, if they wanted to pick out a blue pen, which bag would they choose? Bag S had a blue pen and a white-out in it, and Bag T had a blue pen and a red pen. Most students chose Bag S, while a few did not choose Bag S based on the judgment that each choice had the same chance of picking a blue pen as there was the same number of items in each bag. We interpreted that all of the students had their personal notions of chance, but students choosing Bag S also had an experimental notion of anticipating that they could pick a blue pen every time as they could distinguish a pen from a white-out by touching them in Bag S. After one student really picked out the blue pen in Bag S, the teacher asked whether they were happy with the result. The student answered 'no' as it was not a surprise, and they would have felt more cheerful if they had chosen Bag T. The task let students make a judgment (Judgment), pick one item from the bag they chose

(Enactment), and express their feeling (Reflection). Herein, students had the opportunity to perceive the possible invalidity of an argument, that is, each choice had the same chance of picking a blue pen, and to explain for refutation (critical thinking). The students who did not choose Bag S had the opportunity to regulate their thinking and coordinate personal and experimental notions of chance.

In the second task, the game rules are as follows. There are 10 rounds. In each round, students in the same group take turns playing Roulette A twice, shown in Figure 2, with a fidget spinner. Before each round, students bet a token on one of the outcomes: two Odds, two Evens, one Odd and one Even. If the result was the same as their betting option, they were awarded 1 point. Before playing the game, students were asked whether it was correct that there were three outcomes, so each outcome had a probability of 1/3. Some agreed and some disagreed. Then, they played the game with the curiosity to discover the answer. After the game, the teacher asked the students what they found and felt during the game. Some students said they had more confidence in their judgment that the three outcomes had different probabilities. Some students said they did not change their judgment about the probability of each outcome even though the results seemed to be different from the judgment. The teacher did not give feedback here as a learning opportunity was provided for students to adjust their misconceptions and generate explanations to coordinate their multiple notions. Thus, students were asked to predict what the results of all rounds collected from each group could be and then added the number of each outcome derived from each group. The result showed that the probability of two Odds or two Evens was about 1/4, and the probability of one Odd and one Even was about 1/2. Finally, students were asked to predict the number of one Odd and one Even if 1,000 rounds were played. Most students answered 500 times as the probability of the outcome is about 1/2.

The question asked before the game of the second task (Judgment) was designed by referring to one possible misconception. It offered students the opportunity to perceive the possible invalidity of an argument, and the need to explain for refutation and to evaluate the validity of inferences (critical thinking). During the game, students engaged into playing the Roulette A and were guided to attend to the empirical distribution of the outcomes (Enaction). Then, students were asked to interpret and explain the difference between their predictions and the outcomes (Reflection). By collecting all results from each group, most of them developed a frequentist perspective of probability based on their personal and experimental notions of chance.

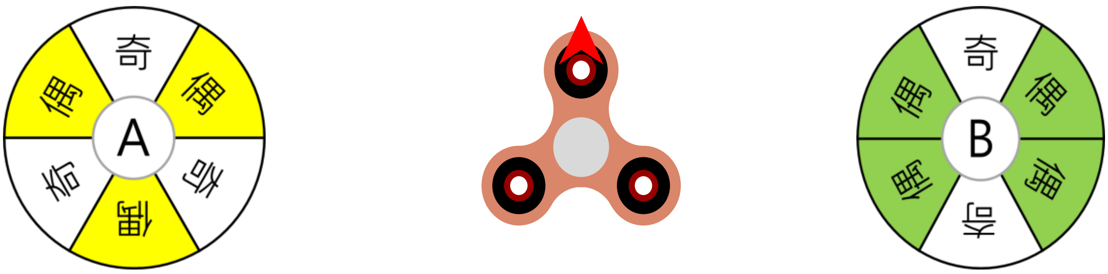

Figure 2. Roulette A, Fidget Sinner, and Roulette B from left to right

The third task aimed to develop the classical perspective of probability using Roulette B, shown in Figure 2. Students were asked to judge all possible outcomes when Roulette B was played once, and all possible outcomes when playing Roulette B twice. The question was to facilitate students' notions of the random situation. Then, they were asked to estimate the probability of two Odds when Roulette B was played twice. The question was to elicit students' personal or theoretical notions of the random situation. A few students displayed the theoretical notion that it was 1/9, as the probability of one Odd was 1/3. Four ways to predict the probability were provided for students to evaluate. The first two ways were based on the results of 10 rounds and all rounds collected from each group when playing Roulette A. The third way was to play Roulette B for 10 rounds, and for the fourth way there was no need to play. Most students immediately declined the first two ways. Nonetheless, some students provided an incomplete explanation that the two Roulettes had different outcomes. After clarifying the meaning, students could regulate their explanation as the probabilities of the same outcome using Roulette A and using Roulette B were different. For the third and fourth ways, students' experimental and theoretical notions were facilitated and coordinated through peer and teacher discussion as some students expressed that both were acceptable. However, some students argued that the number of playing rounds needed to be increased as some extreme result may very likely happen if only 10 rounds were played. At the end of the task, students were asked to predict the number of one Odd and one Even if 900 rounds were played. It was found that students tended to make a prediction based on the experimental notion for Roulette A but based on the theoretical notion for Roulette B.

Judgment was also the first part of the third task. Nonetheless, it involved considering what would happen and aimed to facilitate students' theoretical notions about Roulette B based on reflective abstraction of the experience in playing Roulette A. Although playing Roulette B was not really conducted, most students could base decisions on a thought experiment, which is kind of enaction, to justify their judgment (critical thinking). In the third task, like in the second task, reflection was facilitated by predicting the number of times an outcome would happen and explaining their prediction. During students' engagement in the Judgment-Enaction-Reflection tasks through discussion, their personal, experimental, and theoretical notions were uncovered and coordinated by contrasting and explaining various notions (critical thinking) to develop a frequentist and classical perspective of probability.

The fourth task aimed to provide an opportunity for students to apply theoretical notions flexibly to compare event probabilities for Roulette A and Roulette B and then to make a decision. Students were asked to think about a situation. The class leader can choose Roulette A or B to play with the teacher. If the teacher bet on one of the outcomes and loses, one point would be given to the class leader. 100 points could be redeemed for one McDonald's set. Which Roulette do they suggest the class leader choose? Some students suggested Roulette A, some suggested Roulette B, and some felt no big difference. Then, two questions were asked to help students tackle the problem by providing a hypothetical and numerical premise. One is if Roulette A is played for 180 rounds, how many points could you estimate you get? The other is if Roulette B is played for 180 rounds, how many points could you estimate you get? Some students applied a theoretical notion that the most chance is 1/2 using Roulette A, so the teacher gets about 90 points and the class leader gets the other half. If Roulette B is used, the most chance is 4/9 for the teachers and then the class leader will get about 100 (180x5/9) points. At the same time, students used subjective reasoning, such as 'we win 90 rounds, so it is impossible to eat McDonald'. It indicates students may have theoretical notions but incomplete concept of probability. Other students hypothesized the teacher may not bet on the outcome, which has the largest chance, and reflected that using Roulette B could always benefit them no matter whether the teacher bet on which outcome. The teacher further asked a question, if I let the class leader know I would bet on two Evens, which Roulette would you suggest? It was found that some students could judge correctly that it has more chance to win if Roulette A was used. In the fourth task, students had the opportunity not only to perceive invalid arguments but also to evaluate the validity of hypothetical inferences. Theoretical notions evolved in the fourth task were different from those in the previous tasks. They might be also intertwined with personal (e.g., the use of deterministic words) and experimental (e.g., the assumption of playing 180 rounds) notions, but became the main idea influencing their judgment, prediction and inference. The fourth task moved students one step forward to hypothetical reasoning about an uncertain situation.

DISCUSSION

Previous studies have adopted asymmetric random process, in which equal likelihood for each outcome cannot be assumed, to support students in linking classical and frequentist perspectives of probability through data modeling (Konold et al., 2011; Park & Kim, 2023). In this study, although symmetric random processes were provided for students to develop probabilistic thinking, critical thinking was integrated to design tasks and adopted as an approach to stimulating students' queries about others' or their own judgment. For convincing themselves or others, students' experimental notions of chance spontaneously emerged. Both frequentist and classical perspectives of probability could be applied for making further inference. This analysis shows that a Judgment-Enaction-Reflection learning model, which integrated critical thinking to challenge and co-construct students' personal, experimental, and theoretical notions of chance, is not only feasible to design tasks, but also helpful to understand how students' notions of chance are coordinated to develop subjectivist, frequentist and classical perspectives of probability. Students' theoretical notions, however, were still different from classical perspectives of probability as their inference was also influenced by deterministic thinking.

According to Shaughnessy (1992), four types of stochastic conceptions have been recognized: (1) non-statistical (beliefs, deterministic models, or single-outcome expectations); (2) naïve-statistical (judgmental heuristics or spontaneous understanding of chance); (3) emergent-statistical (normative mathematical models and a distinguishment between intuition and model of chance); and (4) pragmatic-statistical (an in-depth understanding of mathematical models and an ability to compare and contrast different models of chance). The tasks presented in this study have been justified to extend students' responses from naïve-statistical, emergent-statistical, to pragmatic-statistical types. Students' personal, experimental, and theoretical notions of chance were also coordinated in the pragmatic-statistical type.

Students' evaluation and generation of data representations require their critical thinking and creative thinking (Ben-Zvi & Friedlander,1997). In our study, we also found that tasks can stimulate students' structural representations of events, for instance, the use of sample spaces listing (Odd, Even), (Even, Odd), (Odd, Odd), (Even, Even) or a tree diagram, following a critique of 1 Odd 1 Even, 2 Odds, 2 Evens, as a bases for reasoning about the chance situation. This implies critical thinking about notions of sample spaces and their representations is a fruitful approach for coordinating various notions of chance and for constructing appropriate representations which can be used to communicate students' thoughts and support them to explain why some representations are not valid.

CONCLUSION

This study provides a framework, which integrates personal, experimental, and theoretical notions of chance with a Judgment-Enaction-Reflection learning model, for designing tasks from the classical, frequentist, and subjective perspectives. The learning model suggested some important design principles for eliciting, exploring, extending, and coordinating activities as follows:

(1) Judgment tasks include judging the possibility of what will happen and predicting what will happen for stimulating students' possible misinterpretations, and a sub-task of Judgment can be designed with the consideration of what inconsistent personal notions of chance could be.

(2) Enaction tasks include real experiments and thought experiments based on analogous learning experience, and a sub-task of enaction can be designed with the consideration of how to make students feel the need to do experiments.

(3) Reflection tasks include reflecting on feelings and thinking, and a sub-task of reflection can be designed with the consideration of what could be reflected and how to create a need of theoretical notions which can be compared with personal and experimental notions.

The findings showed how a Judgment-Enaction-Reflection learning model works. That is, to uncover different judgments can stimulate students' queries about others' or their own judgments. Enaction, such as manipulating, experimenting or inquiring, can scaffold students to generate examples, counterexamples, data or thought-experiments, which are required to justify the possible misinterpretations or different interpretations. Reflection is an opportunity for students to regulate and coordinate different notions of chances.

The tasks developed in this study provided learning opportunities for students to develop subjectivist, frequentist, and classical perspectives of probability through critically coordinating their personal, experimental, and theoretical notions of chance. Nonetheless, further investigation is needed to justify the effects and efficiency of the learning tasks on critical thinking and probabilistic thinking, and to generalize the learning model to other mathematical topics.

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